Operator Spaces, Linear Logic and the Heisenberg-Schrödinger Duality of Quantum Theory

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Motivation

- Two pictures to describe quantum theory:
  - Schrödinger picture: modify “state”.
  - Heisenberg picture: modify “observable”.
  - Often said to be “dual” to each other.
- Is the Heisenberg-Schrödinger duality also a duality in the sense of Classical Linear Logic (CLL)?

Scientific Approach

- Need a *rigorous* mathematical description of the Heisenberg-Schrödinger duality.
- This can be achieved by combining results from:
  - **Functional analysis**, e.g. Hilbert spaces, Banach spaces – fundamental for quantum theory in general;
  - **Operator algebras**, e.g. von Neumann algebras, C*-algebras – fundamental for the Heisenberg picture (in infinite dimensions);
  - **Noncommutative geometry**, e.g. Operator Spaces – important for both pictures (in infinite dimensions).
- Afterwards, we perform a **categorical** analysis and organise the relevant mathematical structure into models of ILL/CLL.
- Describe facets of the duality in terms of **Polarised** Linear Logic:
  - Schrödinger picture $\Rightarrow$ positive logical polarity;
  - Heisenberg picture $\Rightarrow$ negative logical polarity;

Banach Spaces

Definition

A *Banach space* is a complex vector space X equipped with a norm $\|-\| : X \to [0, \infty)$ (i.e. a normed space) which is complete with respect to the topology induced by the norm. More specifically, this means that every Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ [1] in X has a topological limit, i.e. there exists $x \in X$, such that $\lim_{n\to\infty} \|x_n - x\| = 0$.

Example

- $M_n(\mathbb{C})$, the $n \times n$ complex matrices with operator norm.
- Every finite-dimensional normed space.

[1] $\forall \epsilon > 0. \exists N \in \mathbb{N}. \forall i, j > N. \|x_i - x_j\| < \epsilon$

Bounded Operators

Definition

Let X and Y be Banach spaces and let $f\colon X \to Y$ be a linear map (aka operator). We say that f is:

- *bounded* iff there exists $r > 0$ such that $\|f(x)\| \leq r\|x\|$ for all $x \in X$;
- *continuous* iff f is continuous w.r.t the norm topologies of X and Y;
- a *contraction* iff $\|f(x)\| \leq \|x\|$ for all $x \in X$;
- an *isometry* iff $\|f(x)\| = \|x\|$ for all $x \in X$.

Definition

We write **Ban** for the category of Banach spaces with contractions as morphisms and **FdBan** for the full subcategory consisting of finite-dimensional Banach spaces.

Banach Spaces of Operators

- The space $B(X,Y) \stackrel{\text{def}}{=} \{f\colon X \to Y \mid f \text{ bounded}\}$ is a Banach space w.r.t the *operator norm* given by

$$\|f\| \stackrel{\text{def}}{=} \sup\{\|f(x)\| \ : \ x \in X \text{ and } \|x\| \leq 1\}$$

- We write $B(X) \stackrel{\text{def}}{=} B(X,X)$ for the Banach space of bounded operators on X.
- **Example:** $B(\mathbb{C}^n) \cong M_n(\mathbb{C})$, where $\mathbb{C}^n$ is equipped with the ℓ^2-norm.

Banach Space Duals

- We write $X^* \stackrel{\text{def}}{=} B(X, \mathbb{C})$ for the Banach space *dual* of X.
- **Fact:** For any Banach space X, we have $X \hookrightarrow X^{**}$ isometrically.
- **Fact:** If X finite-dimensional, then $X \cong X^{**}$ isometrically.
- If $f\colon X \to Y$ is bounded, the dual map $f^*\colon Y^* \to X^*$ is called the Banach space adjoint.

Banach Space Tensor Products

- Let X and Y be Banach spaces and consider the algebraic tensor product $X \otimes Y$ (i.e. as vector spaces). Not a Banach space, in general.
- Tensor $X \otimes_\alpha Y$ is a completion of $X \otimes Y$ w.r.t a suitable norm α.
- Then $X \otimes Y \subseteq X \otimes_\alpha Y$ is a dense subset.
- Notable tensor products:
  - Injective tensor product $X \otimes_\epsilon Y$, where ϵ is the smallest reasonable norm.
  - Projective tensor product $X \otimes_\pi Y$, where π is the largest reasonable norm.
- If X and Y are finite-dimensional Banach spaces, then:
  - $X \otimes Y = X \otimes_\alpha Y$ as *vector spaces*.
  - $(X \otimes_\epsilon Y)^* \cong X^* \otimes_\pi Y^*$ isometrically.
  - $(X \otimes_\pi Y)^* \cong X^* \otimes_\epsilon Y^*$ isometrically.

Finite-dimensional Banach Spaces: Categorically and Logically

The category **FdBan** of finite-dimensional Banach spaces and linear contractions:

- is $*$-autonomous and has finite products and coproducts;
- therefore also a model of MALL (multiplicative additive linear logic):
  - multiplicative conjunction $X \otimes Y \stackrel{\text{def}}{=} X \otimes_\pi Y$;
  - multiplicative disjunction $X \parr Y \stackrel{\text{def}}{=} X \otimes_\epsilon Y$;
  - linear negation $X^\perp \stackrel{\text{def}}{=} X^*$;
  - additive conjunction $X \& Y \stackrel{\text{def}}{=} X \oplus^\infty Y$;
  - additive disjunction $X \oplus Y \stackrel{\text{def}}{=} X \oplus^1 Y$;

Categorical and Logical Structure of Banach Spaces

The category **Ban** of Banach spaces and linear contractions:

1. has a symmetric monoidal closed structure:
   1.1 Monoidal product $X \otimes_\pi Y$.
   1.2 Internal hom $B(X, Y)$.
2. is complete (products $\iff$ ℓ^∞-direct sums).
3. is cocomplete (coproducts $\iff$ ℓ^1-direct sums).
4. is locally $\aleph_1$-presentable.
5. forms a model of ILL. In fact, two exponentials:
   5.1 The one induced by the adjunction $\mathbf{Set} \underset{\mathrm{Ball}}{\overset{\ell^1}{\underset{\perp}{\rightleftarrows}}} \mathbf{Ban}$
   5.2 The Lafont exponential (induced by 1, 4 and SAFT).

Remark: Nothing new here, this was already known or easy to deduce!

Banach Spaces Enough?

- Banach spaces **(functional analysis)** are fundamental for our understanding of infinite-dimensional quantum theory.
- However, they are not sufficient. We also need **noncommutative geometry** and **operator algebras**.

Hilbert Spaces

Definition

A *Hilbert space* H is a complex inner-product space such that H is a Banach space w.r.t the ℓ^2-norm $\|h\| \stackrel{\text{def}}{=} \sqrt{\langle h, h\rangle}$.

Example

The Hilbert space $\mathbb{C}^n$ with $\langle x, y\rangle \stackrel{\text{def}}{=} \sum_{i=1}^n \overline{x_i} y_i$.

Example

For any set S, the space

$$\ell^2(S) \stackrel{\text{def}}{=} \left\{ f : S \to \mathbb{C} \mid \sum_{s \in S} |f(s)|^2 < \infty \right\}$$

with inner product $\langle f | g\rangle \stackrel{\text{def}}{=} \sum_{s \in S} \overline{f(s)} g(s)$. Every Hilbert space H is unitarily isomorphic to $\ell^2(S)$ for a set S with $\mathrm{card}(S) = \dim(H)$.

Pure State vs Mixed State Quantum Computation

- In *pure state* quantum computation:
  - A pure state $|\psi\rangle$ is a normalised vector in a Hilbert space H.
  - Emphasis on unitary dynamics, i.e. operations described by unitary operators $U\colon H_1 \to H_2$.
- In *mixed state* quantum computation: emphasis on observational behaviour. Hilbert spaces not enough!

Operators on Hilbert Spaces

- The Hilbert space tensor product $H \overset{2}{\otimes} K$ gives a symmetric monoidal structure.
- No monoidal closed structure for infinite-dimensional Hilbert spaces.
  - $B(H_1, H_2)$ is a Banach space **(functional analysis)**.
  - $B(H_1, H_2)$ is also an operator space **(noncommutative geometry)**.
  - $B(H) \stackrel{\text{def}}{=} B(H, H)$ is a von Neumann algebra **(operator algebra)**.

Are Hilbert spaces enough?

- **Problem:** Quantum measurement – not a morphism between Hilbert spaces.
- Quantum measurement $\Rightarrow$ mixed-state computation $\Rightarrow$ noncommutative geometry and operator algebras.

Mixed-state computation and the Heisenberg-Schrödinger Duality

- Quantum operations (also known as channels) can be modeled in two pictures:
  - Heisenberg picture – as NCPU (normal completely-positive unital maps) $\varphi \colon B(H_2) \to B(H_1)$.
  - Schrödinger picture – as CPTP (completely-positive trace-preserving maps) $\psi \colon T(H_1) \to T(H_2)$.
  - We have $B(H) \cong T(H)^*$ as Banach spaces and $\varphi = \psi^*$ (modulo isomorphism).

Trace Class Operators

- The trace class $T(H) \subseteq B(H)$ is a two-sided ideal for which we have a meaningful notion of trace.
- $T(H)$ is a Banach space w.r.t trace norm and $T(H)^* \cong B(H)$, i.e. $T(H)$ is the predual of $B(H)$.
- We have a bounded operator $\mathrm{tr}\colon T(H) \to \mathbb{C} :: f \mapsto \sum_\lambda \langle e_\lambda, fe_\lambda \rangle$, where $\{e_\lambda\}_{\lambda\in\Lambda}$ can be any ONB of H.
- **Intuition:** Recall that in PLL, for a positive formula P we have $P \multimap !P$ and so we have canonical counit and comultiplication maps.
- For $T(H)$, obvious candidate for counit, namely the trace.

Quantum Operations in the Schrödinger Picture

- An element $t \in T(H)$ is positive if $\langle h, th \rangle \geq 0$, for all $h \in H$.
- A bounded map $\varphi \colon T(H_1) \to T(H_2)$ is:
  - *positive*, if φ preserves positive elements;
  - *completely positive*, if (intuitively) $\varphi \otimes \mathrm{id} \colon T(H_1 \overset{2}{\otimes} \mathbb{C}^n) \to T(H_2 \overset{2}{\otimes} \mathbb{C}^n)$ is positive for every $n \in \mathbb{N}$. Full details omitted.
  - *trace-preserving*, if $\mathrm{tr}(\varphi(t)) = \mathrm{tr}(t)$.
- A *quantum operation (aka quantum channel)* from H_1 to H_2 is a CPTP map $\varphi \colon T(H_1) \to T(H_2)$.
- A state of H in the Schrödinger picture is a density operator: $\rho \in T(H)$ such that $\mathrm{tr}(\rho) = 1$ and $0 \leq \rho$.
- Equivalently, states may be identified with the (C)PTP maps $\varphi \colon \mathbb{C} \to T(H)$.
- **Example:** $\sum_{n=1}^{\infty} 2^{-n} |n\rangle \langle n|$ is a density operator on $\ell^2(\mathbb{N})$.

Normal Maps

- A linear map $\varphi\colon B(H_1) \to B(H_2)$ is called *normal* if there exists a (necessarily unique) bounded operator $\varphi_*\colon T(H_2) \to T(H_1)$ such that

$$\begin{tikzcd}
B(H_1) \arrow[r, "\varphi"] \arrow[d, "\cong"'] & B(H_2) \\
T(H_1)^* \arrow[r, "(\varphi_*)^*"'] & T(H_2)^* \arrow[u, "\cong"']
\end{tikzcd}$$

- **Remark:** Equivalent definition - continuity w.r.t ultraweak topology.
- **Remark:** $B(H)_* \cong T(H)$ and one can see $(-)_*$ as a contravariant functor.

Quantum Operations in the Heisenberg Picture

- A linear map $\varphi\colon B(H_1) \to B(H_2)$ is called *unital* if $\varphi(I_{H_1}) = I_{H_2}$.
- A *quantum operation (aka quantum channel)* from H_1 to H_2 in the Heisenberg picture is a normal completely positive unital (NCPU) map $\varphi\colon B(H_1) \to B(H_2)$.
- A "state" of H in the Heisenberg picture is given by a N(C)PU map $\varphi\colon B(H) \to \mathbb{C}$.
- Every such "state" is necessarily of the form $\varphi = \mathrm{tr}(\rho -)$ for a uniquely determined density operator $\rho \in T(H)$.
- **Remark:** the word "state", as used here, is standard in the *mathematics* literature on operator algebras.

Duality between the Heisenberg and Schrödinger Pictures

- In what sense are the two pictures dual to each other?
- There is a bijective correspondence

$$\mathbf{CPTP}(T(H_1), T(H_2)) \cong \mathbf{NCPU}(B(H_2), B(H_1))$$
$$\varphi \mapsto \varphi^* \qquad \text{(modulo isometric isomorphism)}$$
$$\varphi_* \leftarrow\!\shortmid \varphi \qquad \text{(modulo isometric isomorphism)}$$

- If H_1 and H_2 - finite-dimensional, this is just linear-algebraic transposition.
- General case: transposition in the sense of dual pairings between topological vector spaces. The dual pairing is given by $\mathrm{tr} \colon T(H) \times B(H) \to \mathbb{C} :: (t, a) \mapsto \mathrm{tr}(ta)$.
- This family of bijections determines a contravariant isomorphism of categories $\mathbf{CPTP} \cong \mathbf{NCPU}^{\mathrm{op}}$ where objects are taken to be Hilbert spaces.

Heisenberg-Schrödinger duality = LL duality?

- $T(H)$ and $B(H)$ – not Hilbert spaces in general.
- $T(H)$ and $B(H)$ – Banach spaces, but multiple problems with LL models:
  - Linear-algebraic transposition $(-)^T : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is a positive isometric isomorphism, but not completely positive for $n \geq 2$. How to get rid of it?
  - How to model multiplicative conjunction/disjunction?

$$T(H) \otimes_{?1} T(K) \cong T(H \overset{2}{\otimes} K) \qquad \text{(composition in Schrödinger picture)}$$

$$B(H) \otimes_{?2} B(K) \cong B(H \overset{2}{\otimes} K) \qquad \text{(composition in Heisenberg picture)}$$

- **Solution: noncommutative geometry**. Right kinds of tensors and transposition is not *completely* contractive, not *completely* isometric, not *completely* positive.

Operator Spaces = Noncommutative Banach Spaces

- The noncommutative (i.e. quantum) analogue of Banach spaces are called *Operator Spaces*.
- Part of the program of **noncommutative geometry**.
- $B(H)$ and $T(H)$ are operator spaces.
- Every von Neumann algebra, C*-algebra (from **operator algebra** theory, used for the Heisenberg picture) is also an operator space.
- $(-)^T : M_n \to M_n$ is not a morphism for $n \geq 2$.
- Right kinds of tensor products:

$$T(H) \hat{\otimes} T(K) \cong T(H \overset{2}{\otimes} K) \qquad \text{(completely projective tensor)}$$

$$B(H) \overline{\otimes} B(K) \cong B(H \overset{2}{\otimes} K) \qquad \text{(spatial tensor of von Neumann algebras)}$$

What is an Operator Space?

Definition

An *(abstract) operator space* is a vector space X together with a family of norms

$$\{\|\cdot\|_n : \mathbb{M}_n(X) \to [0, \infty) \mid n \in \mathbb{N}\},$$

such that:

(B) The pair $(\mathbb{M}_1(X), \|\cdot\|_1)$ is a Banach space;

(M1) $\|x \oplus y\|_{m+n} = \max\{\|x\|_m, \|y\|_n\}$

(M2) $\|\alpha x \beta\|_m \leq \|\alpha\|\|x\|_m\|\beta\|$

for each $n, m \in \mathbb{N}$, $x \in \mathbb{M}_m(X)$, $y \in \mathbb{M}_n(X)$, $\alpha, \beta \in M_m$.
We write $M_n(X)$ for the Banach space, in fact operator space, $(\mathbb{M}_n(X), \|\cdot\|_n)$.

What are the morphisms of operator spaces?

Definition

Let $u\colon X \to Y$ be a linear map between operator spaces X and Y. We write $u_n : M_n(X) \to M_n(Y)$ for the linear map $[x_{ij}] \mapsto [u(x_{ij})]$. We say that u is:

- *completely bounded*, if $\|u\|_{\mathrm{cb}} \stackrel{\mathrm{def}}{=} \sup_{n\in\mathbb{N}} \|u_n\| < \infty$.
- a *complete contraction*, if u_n is a contraction for each $n \in \mathbb{N}$.
- a *complete isometry*, if u_n is an isometry for each $n \in \mathbb{N}$.
- a *completely isometric isomorphism*, if u is a surjective complete isometry.

Remark: The map u_n, known as the n-th amplification of u, can be thought of as the map $\mathrm{id}_{M_n} \otimes u$. This can be made precise, via a natural isomorphism, via the completely injective tensor product, which enjoys the property

$$M_n(X) \cong M_n \overset{inj}{\otimes} X$$

Categories of Operator Spaces

- **OS** – the category of operator spaces with *complete* contractions as morphisms.
- **OS** is the noncommutative (or quantum) analogue of **Ban**.
- Noncommutative analogues for Banach space constructions:
  - $CB(X, Y)$ – the operator space of *completely* bounded maps. Internal hom.
  - $X \hat{\otimes} Y$ – the *completely* projective tensor product. Monoidal product.
  - $X \overset{inj}{\otimes} Y$ – the *completely* injective tensor product. Another monoidal product.
  - $X^* \stackrel{\text{def}}{=} CB(X, \mathbb{C})$ – operator space dual. Compatible with Banach space dual.
  - ℓ^1-direct sums and ℓ^∞-direct sums. (Co)products. Compatible with Banach space counterparts.

Finite-dimensional Operator Spaces: Categorically and Logically

The category **FdOS** of finite-dimensional operator spaces and *complete contractions*:

- is $*$-autonomous and has finite products and coproducts;
- therefore also a model of MALL:
  - multiplicative conjunction $X \mathbin{\hat{\otimes}} Y$.
  - multiplicative disjunction $X \overset{inj}{\otimes} Y$.
  - linear negation $X^{\perp} \overset{\mathrm{def}}{=} X^{*}$.
  - additive conjunction $X \& Y \overset{\mathrm{def}}{=} X \oplus^{\infty} Y$.
  - additive disjunction $X \oplus Y \overset{\mathrm{def}}{=} X \oplus^{1} Y$.
- In addition to **Ban**, we also get the right tensors:
  - $T(H_1) \mathbin{\hat{\otimes}} T(H_2) \cong T(H_1 \otimes H_2)$ (composition in Schrödinger picture).
  - $B(H_1) \overset{inj}{\otimes} B(H_2) \cong B(H_1 \otimes H_2)$ (composition in Heisenberg picture).
  - Note: $\overline{\otimes} = \overset{inj}{\otimes}$ for finite-dimensional von Neumann algebras, like above.
- Part of internship project of Thea Li.

Categorical and Logical Structure of Operator Spaces

The category **OS** of operator spaces and *complete* contractions:

1. has a symmetric monoidal closed structure:
   1.1 Monoidal product $X \hat{\otimes} Y$. Moreover, $T(H_1) \hat{\otimes} T(H_2) \cong T(H_1 \overset{2}{\otimes} H_2)$.
   1.2 Internal hom $CB(X, Y)$.
2. is complete (products $\iff$ ℓ^∞-direct sums).
3. is cocomplete (coproducts $\iff$ ℓ^1-direct sums).
4. is locally $\aleph_1$-presentable. (Most important result in our paper)
5. forms a model of ILL. In fact, two exponentials:
   5.1 The one induced by the adjunction $\mathbf{Set} \underset{\text{Ball}}{\overset{\ell^1}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathbf{OS}$
   5.2 The Lafont exponential (induced by 1, 4 and SAFT).

Remark: Principle categorical difference with **Ban**: the *strong* generators in 4.

Mixed-State Computation

- Let $\varphi\colon B(H_1) \to B(H_2)$ be a linear unital map. Then, φ is completely-positive iff φ is a complete contraction.[2]
- Let $\varphi\colon T(H_1) \to T(H_2)$ be a linear trace-preserving map. Then, φ is completely-positive iff φ is a complete contraction.
- Therefore CPTP and NCPU maps are complete contractions and in **OS**.

[2] Also holds for von Neumann algebras and operator systems.

CLL and Heisenberg-Schrödinger Duality

- **OS** is not $*$-autonomous.
- However, $\mathbf{Q} \stackrel{\text{def}}{=} \mathbf{Chu}(\mathbf{OS}, \mathbb{C})$ is (co)complete, $*$-autonomous, has a Lafont exponential and it is a model of CLL. Follows using results of Barr.
- Objects are triples (X, Y, d), where $d \colon X \hat{\otimes} Y \to \mathbb{C}$ is a complete contraction.
- a morphism is a pair $(f, g) \colon (X_1, Y_1, d_1) \to (X_2, Y_2, d_2)$ such that

$$\begin{tikzcd} X_1 \hat{\otimes} Y_2 \arrow[r, "X_1 \hat{\otimes} g"] \arrow[d, "f \hat{\otimes} Y_2"'] & X_1 \hat{\otimes} Y_1 \arrow[d, "d_1"] \\ X_2 \hat{\otimes} Y_2 \arrow[r, "d_2"'] & \mathbb{C} \end{tikzcd}$$

If $X_i = T(H_i)$, $Y_i = B(H_i)$, and $d_i = \mathrm{tr}$, the diagram commutes iff $f = g^t$, i.e. f is the transpose w.r.t dual pairing of the Heisenberg-Schrödinger duality.

Multiplicatives in $\mathbf{Q}$ and the Heisenberg-Schrödinger Duality

Theorem

The monoidal product

$$(T(H_1), B(H_1), \mathrm{tr}) \otimes (T(H_2), B(H_2), \mathrm{tr})$$

in $\mathbf{Q}$ *is the object*

$$(T(H_1) \hat{\otimes} T(H_2), B(H_1) \overline{\otimes} B(H_2), \mathrm{tr}').$$

Remark: We can recover $\overline{\otimes}$ (composition in the Heisenberg picture) from $\hat{\otimes}$ (composition in the Schrödinger picture) via the Chu construction (semantics of LL) applied to **OS**.

PLL and the Heisenberg-Schrödinger Duality

Schrödinger Picture	**Q**	LL_+
System description	$T(H_P)$	P
Quantum composition	$T(H_P) \hat{\otimes} T(H_R) \cong T(H_P \overset{2}{\otimes} H_R)$	$P \otimes R$
Classical composition	$T(H_P) \overset{1}{\oplus} T(H_R)$	$P \oplus R$

Heisenberg Picture	**Q**	LL_-
System description	$B(H_N)$	N
Quantum composition	$B(H_N) \overline{\otimes} B(H_M) \cong B(H_N \overset{2}{\otimes} H_M)$	$N \parr M$
Classical composition	$B(H_N) \overset{\infty}{\oplus} B(H_M)$	$N \,\&\, M$

Schrödinger picture $\Rightarrow$ positive logical polarity

Heisenberg picture $\Rightarrow$ negative logical polarity

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Heisenberg Picture	**Q**	LL_-
System description	$B(H_N)$	N
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Remark: can be extended to von Neumann algebras and their preduals.

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Remark: can be extended to von Neumann algebras and their preduals.
Future work: constructive description of Lafont exponential in operator space theory.

Future Work: Quantum Coherence Spaces

- Quantum Coherence Spaces à la Girard:
  - not appropriate from a quantum point of view (e.g. criticism by Selinger).
  - main problem: positivity instead of *complete* positivity, i.e., geometry instead of *noncommutative* geometry.
- This paper: we lay the groundwork for future work on quantum coherence spaces.

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- **Future work:** QCS such that we restrict relevant homsets to CPTP/NCPU (or CPTNI/NCPSU for recursion).
- More results in our paper (arXiv:2505.06069):
  - categorical structure of **OS**;
  - interaction between pure and mixed state quantum computation;
  - Haagerup tensor and links to BV-logic?

Thank you for your attention!