Linearization *via* Rewriting

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Summary

A Tale of Two Cities

Qualitative approximation

Question

What is computed?

Information approximation
(Scott continuity)

Quantitative approximation

Question

How many resources are used?
(time, space . . .)

Resource approximation
(Taylor expansion)

Linear Approximation of λ-Terms

Approximation relation

$$\frac{}{x \lhd x} \qquad \frac{s \lhd M}{\lambda x.s \lhd \lambda x.M}$$

$$\frac{s \lhd M \qquad \vec{t} \lhd N}{s\vec{t} \lhd MN} \qquad \frac{t_1 \lhd M \quad \cdots \quad t_k \lhd M}{\langle t_1, \ldots, t_k \rangle \lhd M}$$

$s \lhd M$ iff $s \in \mathcal{T}(M)$.

We have the **linear application**:

$$s\langle t_1, \ldots, t_k \rangle \qquad s : \langle a_1, \ldots, a_k \rangle \multimap b \qquad t_i : a_i$$

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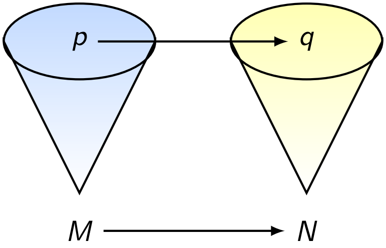
Simulating the Evaluation

Theorem (Simulation)

Given $M \rightsquigarrow N$, there exists $p \lhd M$ and $q \lhd N$ s.t. $p \rightsquigarrow q$.



Theorem (Termination)

If there exists $s \in \mathcal{T}(M)$ that is typable, then the head-reduction of M terminates.

Let $\mathrm{ev}(M)$ be the *number* of evaluation step needed by M to compute a result.

Theorem (De Carvalho 2007)

The following statements hold.

- *If $s \in \mathcal{T}(M)$ and it is typable, then* $\mathrm{ev}(M) \leq \mathrm{size}(s)$.
- *There exists $s \in \mathcal{T}(M)$ that is typable and* $\mathrm{ev}(M) = \mathrm{size}(s)$.

Termination and Quantitative Bounds

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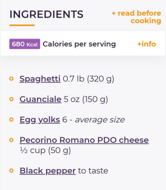
Question

How do we *compute* well-behaved approximations?

Extensional Collapse (Ehrhard 2012)

→ Orthogonality →



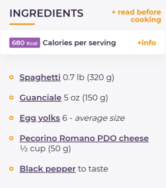
- Powerful categorical approach.
- Implicit and not effective.

Question

How do we *compute* well-behaved approximations?

Our answer (Dal Lago and O. 2025)

→ structural reduction →



- effective translation (**algorithm** given by **rewriting**).
- Explicit presentation of a 2-categorical construction.
- Embedding of qualitative type system.

Summary

Running Example

Consider

$$M = \lambda z^{o \Rightarrow o}.z(zx) \qquad N = \lambda w^{o}.vww$$

We can type

$$x : o \vdash M : (o \Rightarrow o) \Rightarrow o \qquad v : o \Rightarrow o \Rightarrow o \vdash N : o \Rightarrow o$$

Hence

$$x : o, v : o \Rightarrow o \Rightarrow o \vdash MN : o$$

A Naive Attempt

We try to linearize MN, working *inductively*.
Then we get

$$\mathrm{lin}(M) = \lambda z^{\langle \langle o \rangle \multimap o, \langle o \rangle \multimap o \rangle}. z \langle z \langle x \rangle \rangle \qquad \mathrm{lin}(N) = \lambda w^{\langle o, o \rangle}. v \langle w \rangle \langle w \rangle$$

with typing:

$$x : \langle o \rangle \vdash \mathrm{lin}(M) : \langle \langle o \rangle \multimap o, \langle o \rangle \multimap o \rangle \multimap o$$

and

$$v : \langle \langle o \rangle \multimap \langle o \rangle \multimap o \rangle \vdash \mathrm{lin}(N) : \langle o, o \rangle \multimap o$$

Lack of Communication

$\mathrm{lin}(M) : \langle\langle o \rangle \multimap o\rangle^2 \multimap o$ $\quad\quad$ $\mathrm{lin}(N) : \langle o, o \rangle \multimap o$

!

However, we have a *nested* ‘structural rule’:

$$c_o \multimap o : (\langle o, o \rangle \multimap o) \to (\langle o \rangle \multimap o)$$

where $c_o : \langle o \rangle \to \langle o, o \rangle$.

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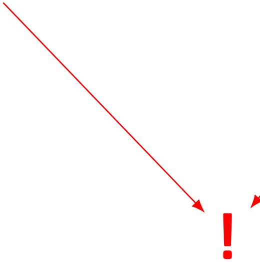

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Summary

Resource Types

Grammar of Types

$$\mathrm{RT} \ni a, b ::= o \mid \vec{a} \multimap b \qquad \vec{a} ::= \langle a_1, \ldots, a_k \rangle \qquad (k \in \mathbb{N})$$

Morphisms of Types

$$\frac{}{id_o : o \to o} \qquad \frac{\langle \alpha; \vec{f} \rangle : \vec{b} \to \vec{a} \qquad f : a \to b}{\langle \alpha; \vec{f} \rangle \multimap f : (\vec{a} \multimap a) \to (\vec{b} \multimap b)}$$

$$\frac{\alpha : [k] \to [l] \qquad f_1 : a_{\alpha(1)} \to b_1 \quad \cdots \quad f_k : a_{\alpha(k)} \to b_k}{\langle\, \alpha\, ;\ f_1, \ldots, f_k \,\rangle : \langle a_1, \ldots, a_l \rangle \to \langle b_1, \ldots, b_k \rangle}$$

Morphisms as Operations on Resources

Resource Types and morphisms induce a *category*. If one restrict to

$$\mathsf{Int}(\mathsf{RT}) = \{\langle a_1, \dots, a_k \rangle \mid a_i \in \mathsf{RT}\}$$

we have a *cartesian category*, where the product is list concatenation.

Examples

$$\langle\, \sigma : [2] \cong [2]\,;\, id_a, id_b \rangle : \langle a, b \rangle \to \langle b, a \rangle \qquad \text{(permutations)}$$

$$\pi_i = \langle\, \iota_i : [1] \subseteq [2]\,;\, id_{a_i} \rangle : \langle a_1, a_2 \rangle \to a_i \qquad \text{(projections)}$$

$$\mathsf{c}_a = \langle\, \mathsf{d} : [2] \to [1]\,;\, id_a, id_a \rangle : \langle a \rangle \to \langle a, a \rangle \qquad \text{(diagonals)}$$

$$\mathsf{c}_o \multimap id_o : (\langle o, o \rangle \multimap o) \to (\langle o \rangle \multimap o) \qquad \text{(nested structural rule)}$$

Structural Resource Terms

Grammar of Terms

$$\Lambda_{lin} \ni s, t ::= x \mid \lambda x^{f}.s \mid s\vec{t} \qquad \vec{t} ::= \langle s_1, \ldots, s_k \rangle \qquad (k \in \mathbb{N})$$

where f is a resource morphism.

Typing Context Formation

$$\frac{}{() \text{ ctx}} \qquad \frac{\gamma \text{ ctx} \quad x \text{ fresh} \quad \vec{a} \text{ int type}}{\gamma, x : \vec{a} \text{ ctx}}$$

For $\gamma = x_1 : \vec{a}_1, \ldots, x_n : \vec{a}_n$ and $\delta = x_1 : \vec{b}_1, \ldots, x_n : \vec{b}_n$ we set

$$\gamma \otimes \delta = x_1 : \vec{a}_1 :: \vec{b}_1, \ldots, x_n : \vec{a}_n :: \vec{b}_n$$

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Type System

Rules

$$\frac{}{x : \langle a \rangle \vdash x : a} \qquad \frac{\gamma, x : \vec{a} \vdash s : b \qquad f : \vec{b} \to \vec{a}}{\gamma \vdash \lambda x^f . s : \vec{b} \multimap b}$$

$$\frac{\gamma_0 \vdash s : \vec{a} \multimap b \qquad \gamma_1 \vdash \vec{t} : \vec{a}}{\gamma_0 \otimes \gamma_1 \vdash s\vec{t} : b} \qquad \frac{\gamma_1 \vdash t_1 : a_1 \quad \cdots \quad \gamma_k \vdash t_k : a_k}{\gamma_1 \otimes \cdots \otimes \gamma_k \vdash \langle t_1, \ldots, t_k \rangle : \langle a_1, \ldots, a_k \rangle}$$

Relevant typing context.

Uniqueness of typing derivation.

Encoding our Favourite Term

Iterators

$$\dfrac{\dfrac{}{x : \langle\rangle, z : \langle o \rangle \multimap o} \quad \dfrac{\dfrac{}{x : \langle\rangle, z : \langle o \rangle \multimap o \vdash z : \langle o \rangle \multimap o} \quad \dfrac{}{x : \langle o \rangle, z : \langle\rangle \vdash x : o}}{x : \langle o \rangle, z : \langle \langle o \rangle \multimap o \rangle \vdash z\langle x \rangle : o}}{x : \langle o \rangle, z : \langle \langle o \rangle \multimap o, \langle o \rangle \multimap o \rangle \vdash z\langle z \langle x \rangle \rangle : o} \quad c_{\langle o \rangle \multimap o} : \langle \langle o \rangle \multimap o \rangle \to \langle \langle o \rangle \multimap o \rangle^2$$

$$\overline{x : \langle o \rangle \vdash \lambda z^{c_{\langle o \rangle \multimap o}}.z\langle z \langle x \rangle \rangle : \langle \langle o \rangle \multimap o \rangle \multimap o}$$

We can type

$$x : \langle o \rangle, v : \langle \langle o \rangle \multimap \langle o \rangle \multimap o \rangle \vdash (\lambda z^{c_{\langle o \rangle \multimap o}}.z \langle z \langle x \rangle \rangle) \langle \lambda w^{c_o}. v \langle w \rangle \langle w \rangle \rangle : o$$

Action of Morphisms

$$\pi$$
$$\vdots$$
$$\gamma, x : \vec{a} \vdash s : b$$
$$f \;:\; \vec{b} \;\to\; \vec{a}$$

$$\pi\{f/x\}$$
$$\vdots$$
$$\gamma^{[\nu]}, x : \vec{b}^{[\nu]} \vdash s\{f/x\} : b$$
$$\nu \;:\; (\gamma, x : \vec{b}) \;\to\; (\gamma^{[\nu]}, x : \vec{b}^{[\nu_x]})$$

$$\pi$$
$$\vdots$$
$$\gamma \vdash s : b$$
$$g \;:\; b \;\to\; c$$

$$[g]\,\pi$$
$$\vdots$$
$$\gamma^{[\mu]} \vdash [g]\,s : c$$
$$\mu \;:\; \gamma \;\to\; \gamma^{[\mu]}$$

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Acting on Terms

Coherent Operation

Let $s \;=\; x\,\langle\, y\,\langle x\rangle\,\rangle$ with

$$x : \langle\,\langle o\rangle \multimap o, o\,\rangle\,,\; y : \langle\,\langle o\rangle \multimap o\,\rangle$$

and

$$f \;=\; \langle\, id\,;\, c_o \multimap o,\; o\,\rangle$$

then we have

$$s\{f/x\} \;=\; x\,\langle\, y\,\langle x\rangle\,,\; y\,\langle x\rangle\,\rangle$$

with

$$x : \langle\,\langle o\rangle \multimap o, o, o\,\rangle\,,\; y : \langle\,\langle o\rangle \multimap o, \langle o\rangle \multimap o\,\rangle$$

Exponential and Linear Reductions

LTS with

$$(\gamma \vdash s : a) \ \to_{\theta;f} (\gamma' \vdash s' : a') \qquad \theta : \gamma \to \gamma' \qquad f : a' \to a$$

Reduction Rules (Ground Steps)

$$(\gamma \vdash \lambda x^{f}.s : \vec{a} \multimap b) \ \to^{e}_{\nu;\nu_x \multimap b} \ (\gamma^{[\nu]} \vdash \lambda x^{\vec{a}^{[\nu_x]}}.s\{f/x\} : \vec{a}^{[\nu_x]} \multimap b)$$

$$(\gamma \otimes \delta \vdash (\lambda x.s)\vec{t} : b) \ \to^{l}_{\sigma;b} \ ((\gamma \otimes \delta)^{[\sigma]} \vdash s\{\vec{t}/x\} : b)$$

Reduction Rules (Interaction)

$$\frac{(\gamma \vdash s : \vec{a} \multimap b) \ \to_{\theta;f \multimap g} \ (\gamma' \vdash s' : \vec{a}' \multimap b')}{(\gamma \otimes \delta \vdash s\vec{t} : b) \ \to_{\theta \otimes \mu} \ (\gamma' \otimes \delta^{[\mu]} \vdash s'([f]\vec{t}) : b')}$$

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Termination and Confluence

Theorem

Exponential and linear reductions terminate.

- **Logical relations** $\rightsquigarrow$ exponential.
- **Decreasing size** $\rightsquigarrow$ linear.

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$$(\lambda z^{c_{\langle o\rangle \multimap o}}.z\,\langle\, z\,\langle x\rangle\,\rangle)\,\langle\, \lambda z^{c_o}.\,v\,\langle z\rangle\,\langle z\rangle\,\rangle \;\to_{v:c;id}\; (\lambda z^{\langle\langle o\rangle \multimap o\rangle^2}.z\,\langle\, z\,\langle x\rangle\,\rangle)\,\langle\, \lambda z^{c_o}.\,v\,\langle z\rangle\,\langle z\rangle\,\rangle^2$$

by the exponential step

$$\lambda z^{c_{\langle o\rangle \multimap o}}.z\,\langle\, z\,\langle x\rangle\,\rangle \;\to_{id;c_{\langle o\rangle \multimap o}\multimap o}\; \lambda z^{\langle\langle o\rangle \multimap o\rangle^2}.z\,\langle\, z\,\langle x\rangle\,\rangle$$

and

$$(\lambda z^{\langle\langle o\rangle \multimap o\rangle^2}.z\,\langle\, z\,\langle x\rangle\,\rangle)\,\langle\, \lambda z^{c_o}.\,v\,\langle z\rangle\,\langle z\rangle\,\rangle^2 \to_{x:c^4;id} (\lambda z^{\langle\langle o\rangle \multimap o\rangle^2}.z\,\langle\, z\,\langle x\rangle^2\,\rangle^2)\,\langle\, \lambda z^{\langle o\rangle^2}.\,v\,\langle z\rangle\,\langle z\rangle\,\rangle^2$$

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Everything's on the Table

$$(\lambda z^{c_{\langle o\rangle \multimap o}}.z\,\langle\, z\,\langle x\rangle\,\rangle)\,\langle\, \lambda z^{c_o}.\,v\,\langle z\rangle\,\langle z\rangle\,\rangle \;\rightarrow_{v:c;id}\; (\lambda z^{\langle\langle o\rangle \multimap o\rangle^2}.z\,\langle\, z\,\langle x\rangle\,\rangle)\,\langle\, \lambda z^{c_o}.\,v\,\langle z\rangle\,\langle z\rangle\,\rangle^2$$

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Summary

Linearizing Type Systems

- Simply typed terms and idempotent intersection type derivations can be embedded in the structural resource calculus.
- The embedding preserve and *factorize* the β-reduction:

$$M \to N \qquad \mathrm{struct}(M) \to^{e} t \to^{l} \mathrm{struct}(N)$$

  for some structural term t.
- We can then *prove* computational properties of type systems exploiting their linearizations.

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Collapsing Intersection Types, Operationally

- We can compute both *cartesian* (aka Scott) and *linear* relational semantics by exploiting the structural resource calculus:

$$[\![M]\!]_{\mathrm{scott}} = \{(\gamma; a) \mid \text{ there exists } s \lhd M, \gamma \vdash s : a\}$$

$$[\![M]\!]_{\mathrm{lin}} = \{(\gamma; a) \mid \text{ there exists } s \lhd M, \gamma \vdash_{lin} s : a\}$$

- *Proof-relevant* extensional collapse:

$$\mathrm{nf}(-)^{e} : \mathcal{T}_{\mathrm{cart}}(M)(\gamma; a) \to \sum_{\delta; b} \mathcal{T}(M)_{\mathrm{lin}}(\delta; b) \times \mathrm{RT}(b, a) \times \mathrm{Ctx}(\delta, \gamma)$$

from which one can prove Ehrhard's collapse

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Conclusions

Many perspectives

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- Abstract machine for the exponential reduction.
- Modular method to establish complexity bounds.
- Gödel's Koan? ('canonical' combinatorial proof of termination).
- Proof-relevant collapse as a 2-dimensional categorical construction.
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- Abstract machine for the exponential reduction.
- Modular method to establish complexity bounds.
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Thank You!