Realizability models of set theory

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Realizability

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We establish a correspondence between formulae and programs in a way that is compatible with the rules of deduction.

For instance, a realizer of an implication $A \Rightarrow B$ is a program which, whenever applied to a realizer of A, returns a realizer of B.

From Curry Howard to Forcing

Logical system

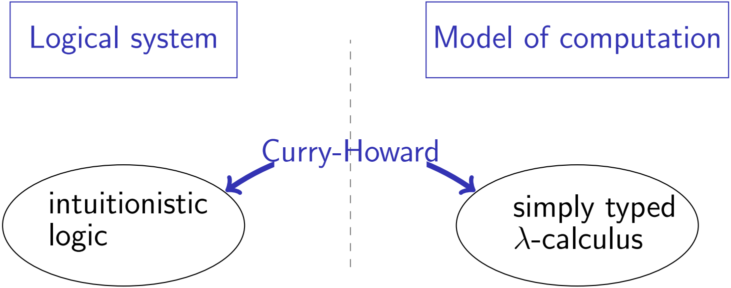
Model of computation

Kleene's Realisability

Heyting Arithmetic

recursive functions

From Curry Howard to Forcing



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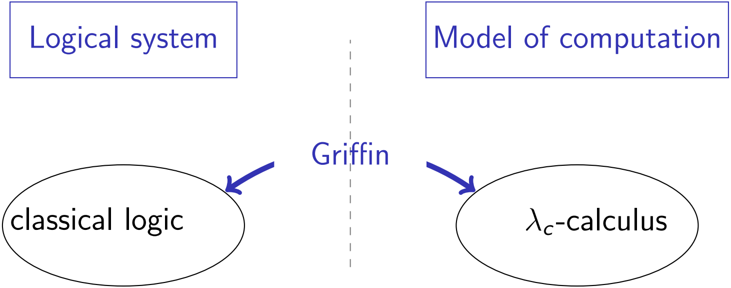

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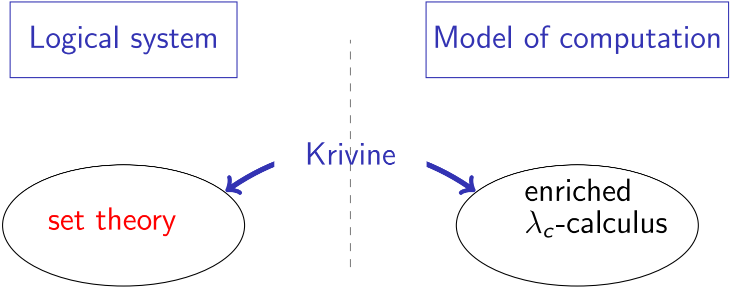


Forcing in a nutshell

Cohen 1963: Forcing is the main technique for defining models of ZF(C) and proving independence and relative consistency results.

We consider a Boolean algebra $(\mathbb{B}, 0, 1, \wedge_{\mathbb{B}} \vee_{\mathbb{B}}, \neg_{\mathbb{B}})$, and we "evaluate" each formula of set theory using the elements of $\mathbb{B}$:

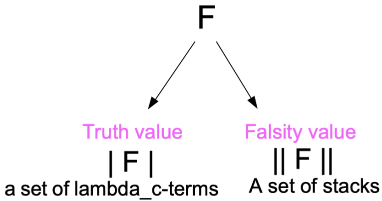
$$\varphi \mapsto ||\varphi|| \in \mathbb{B}$$

$$||\varphi \wedge \psi|| = ||\varphi|| \wedge_{\mathbb{B}} ||\psi||$$
$$||\varphi \vee \psi|| = ||\varphi|| \vee_{\mathbb{B}} ||\psi||$$
$$||\neg\varphi|| = \neg_{\mathbb{B}} ||\varphi||$$

$p \Vdash \varphi$ ("p forces φ") when $p \leq ||\varphi||$

Given an ultrafilter U on $\mathbb{B}$, the $\text{Theory} := \{\varphi : \exists p \in U(p \Vdash \varphi)\}$ forms a coherent classical theory which contains ZF(C) (if we do the process starting from a model of ZF(C), the *ground model*)

Classical realizability in a nutshell



A program $p \Vdash F$ ("p realizes F") when it is in the truth value of F, that is when p is 'orthogonal' to every stack π in the falsity value of F,

meaning that the process $p * \pi \in \perp\!\!\!\perp$, where $\perp\!\!\!\perp$ is the so-called 'Pole'.

Classical realizability in a nutshell

Then we choose some privileged λ_c-terms that we call *realizers*/*proof-like terms*, here denoted $\mathcal{PL}$.

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We show that T forms a coherent theory, which is closed by classical natural deduction and under the right constraints it will contain the axioms we want to realize (e.g. ZF); a realizability model is a model of such a theory.

Realizability algebras

The main ingredients are :

- Λ a set of programs (λ_c-terms$+$ ...)
- Π a set of stacks
- $\mathcal{PL}$ a set of realizers/proof-like terms
- $\prec_K$ the execution, a pre-order on processes $t * \pi$ (where t is a program and π is a stack)
- $\perp\!\!\!\perp$ the pole a set of processes $\prec_K$-upward closed (it defines the "orthogonal processes")

Terms and stacks

$\Lambda^{\text{open}}_{A,B}$ (λ_c-**terms**) :

$$
\begin{array}{lcll}
t, u & ::= & x & \text{(variable; we choose a countable set of variables} \\
 & \mid & tu & \text{(application)} \\
 & \mid & \lambda x.t & \text{(abstraction; } x \text{ is a variable and } t \text{ is a } \lambda_c\text{-term)} \\
 & \mid & \text{æ} & \text{(call-with-current-continuation)} \\
 & \mid & k_\pi & \text{(continuation constants; } \pi \text{ is a stack)} \\
 & \mid & \xi_\alpha & \text{(special terms; } \alpha \in A\text{)}
\end{array}
$$

$\Pi_{A,B}$ (**Stacks**) :

$$
\begin{array}{lcll}
\pi & ::= & \omega_\beta & \text{(stack bottoms; } \beta \in B\text{)} \\
 & \mid & t \bullet \pi & \text{(} t \text{ is a closed } \lambda_c\text{-term and } \pi \text{ is a stack)}
\end{array}
$$

The execution

$\prec_K$ is the smallest preorder on the set of processes such that

$$
\begin{array}{rcll}
tu * \pi & \succ_K & t * u \bullet \pi & \text{(push)} \\
\lambda x.t * u \bullet \pi & \succ_K & t[x := u] * \pi & \text{(grab)} \\
cc * t \bullet \pi & \succ_K & t * k_\pi \bullet \pi & \text{(save)} \\
k_{\pi'} * t \bullet \pi & \succ_K & t * \pi' & \text{(restore)}
\end{array}
$$

Classical realizability

No stack is in $||\top||$, every stack is in $||\bot||$
$||\varphi \Rightarrow \psi|| = \{\xi \,\centerdot\, \pi;\ \xi \in |\varphi| \text{ and } \pi \in ||\psi||\}$

$\xi \Vdash \varphi$ (also denoted $\xi \in |\varphi|$) if $\xi * \pi \in \perp\!\!\!\perp$ for every stack in $||\varphi||$

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call-cc realizes Peirce's law $(((A \Rightarrow B) \Rightarrow A) \Rightarrow A)$

Non extensional set theory

In order to realize the axioms of set theory, we work with a *non-extensional* version of ZF, called ZF_ε, with...

two membership relations:

- $\in$ the usual extensional one
- ε a (strict) non-extensional one

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two membership relations:

- $\in$ the usual extensional one
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... and two equality relations

- $\simeq$ the usual extensional one
- $=$ Leibniz identity

The realizability relation

We define $|\varphi|$ and $||\varphi||$ simultaneously and by induction on φ in the language of realizability.

- no stack is in $||\top||$, every stack is in $||\bot||$
- $||a \not\varepsilon b|| = \{\pi \text{ stack } ; \ (a, \pi) \in b\}$
- $||a \subseteq b|| = \{\xi \centerdot \pi; \ \exists c (c, \pi) \in a \text{ and } \xi \Vdash c \notin b\}$
- $||a \notin b|| = \{\xi \centerdot \xi' \centerdot \pi; \ \exists c (c, \pi) \in b \text{ and } \xi \Vdash a \subseteq c \text{ and } \xi' \Vdash c \subseteq a\}$
- $||\varphi \Rightarrow \psi|| = \{\xi \centerdot \pi; \ \xi \Vdash \varphi \text{ and } \pi \in ||\psi||\}$
- $||\forall x \varphi|| = \bigcup_a ||\varphi[a/x]||$

$\xi \in |\varphi| \iff \forall \pi \in ||\varphi|| (\xi \star \pi \in \perp\!\!\!\perp)$

$\xi \Vdash \varphi$ means $\xi \in |\varphi|$

Realizability models for set theory

ZF_ε is a conservative extension of ZF. The realizability theory contains the axioms of ZF_ε (provided ZF is consistent).

Any model of such a theory yields a structure (actually many) in the language of ZF_ε denoted $\mathcal{N}_\varepsilon$ or $\mathcal{N}$, and a structure (actually many) in the language of ZF, denoted $\mathcal{N}_\in$.

Realizability models for set theory

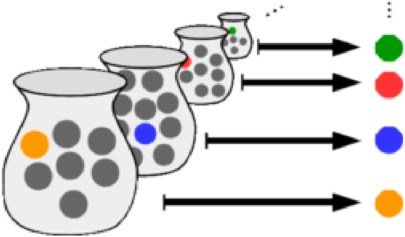
Thus the axioms of set theory are realizable.

Exemple:

- the Pairing axiom is realized by $\lambda x.((xI)I)$,
- the Axiom of Foundation is realized by the fixed point combinator,
- Dependent choice can be realized by quote/a clock...

The Axiom of Choice

What is the computational content
of the axiom of choice?

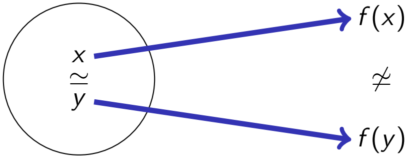


Non extensional axiom of choice

We can “easily” realize a non extensional version of AC, called NEAC

NEAC: existence of a *non-extensional* choice function, i.e.

$$\begin{array}{rcl} x = y & \Rightarrow & f(x) = f(y), \text{ but} \\ x \simeq y & \not\Rightarrow & f(x) \simeq f(y) \end{array}$$



Realizing dependent choice

Krivine 2004

Dependent Choice can be realized using quote/a clock and or the bar recursion.

$$quote * t.s.\pi \succ t * \underline{n}_s.\pi$$

where $s \mapsto n_s$ is some fixed enumeration of Λ in order-type ω.

Realizing the axiom of choice

$\mathbf{ZL}_\kappa$: Let X be a non-empty set, and let R be a binary relation on X such that for every $\alpha < \kappa$, every R-chain $s = (s_\beta)_{\beta<\alpha}$ of length α can be extended (i.e. one can find an element $y \in X$ such that $s_\beta \; R \; y$ for every $\beta < \alpha$), then there is an R-chain of length κ.

$$AC \iff \forall \kappa \in Ord \; (ZL_\kappa)$$

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Fontanella Geoffroy 2021

For every cardinal κ in a model of ZFC, we can construct a realizability model of $ZF + ZL_{\hat{\kappa}}$ (using a "generalized quote" denoted χ).

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Krivine 2021

The full Axiom of Choice can be realized (but no explicit realizer).

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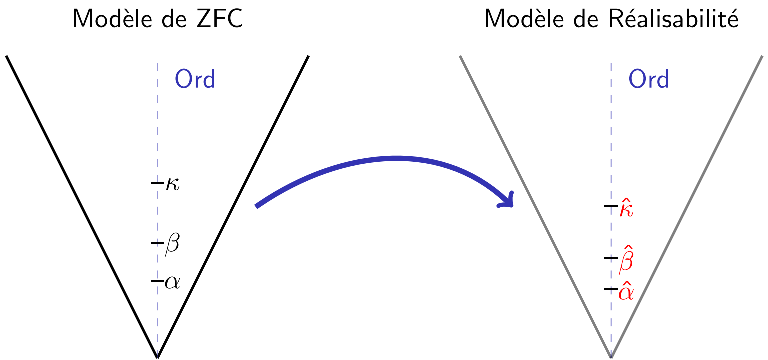

Fontanella Geoffroy 2021

Starting from a model $\mathcal{M}$ of ZF + global choice we can define for every cardinal κ of $\mathcal{M}$ a realizability model where κ has a representative $\hat{\kappa}$ such that $ZF + ZL_{\hat{\kappa}}$ is realized

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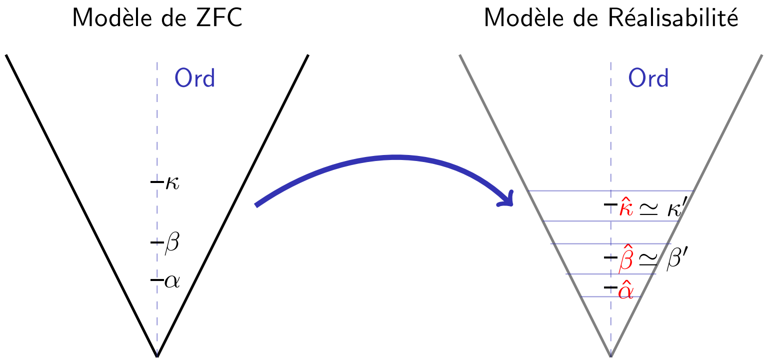

Sketch - the representatives

- We consider a calculus with κ many terms
- We add an instruction χ which "compares the terms by their ordinal index" (similar to quote)
- we define for every ordinal $\alpha \leq \kappa$ in the ground model, a set $\hat{\alpha}$
- We show that "hat ordinals" "represent" their counterpart in the ground model.

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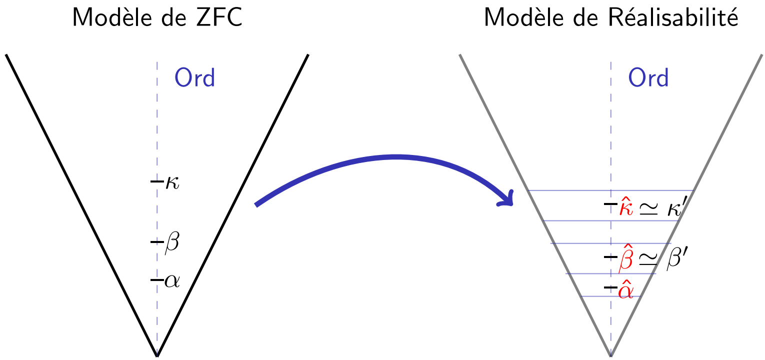

Sketch - realizing $DC_{\hat{\kappa}}$

- $\hat{\kappa}$ has a $=$-unique representative of each of its $\simeq$-classes of elements
- NEAC entails a choice function over the representatives
- we assign the same value to the other elements in the same class
- we realize $ZL_{\hat{\kappa}}$

Fontanella Geoffroy 2021

We fix an enumeration of the terms $\Lambda = \{\nu_\alpha\}_{\alpha<\kappa}$.

For every $\alpha \leq \kappa$,

$$\hat{\alpha} := \{(\hat{\beta}, \nu_\beta \bullet \pi) | \pi \in \Pi, \beta < \alpha\}$$

Proposition

For every $\alpha \leq \kappa$, we can realize (by a proof-like term) that $\hat{\alpha}$ is an ε-ordinal.

Fontanella Matthews 2024

Over ZF, equivalent definitions of ordinals:

1. An ordinal is a transitive set which is well-ordered by $\in$.
2. An ordinal is a transitive set a satisfying trichotomy. That is, for any $x, y, \in a$, precisely one of the following hold: $x \in y$, $y \in x$, or $x = y$.
3. An ordinal is a transitive set of transitive sets.

Fontanella Matthews 2024

In $\mathrm{ZF}_{\varepsilon}$ these notions are not equivalent.

1. an ε-transitive set which is ε-well-ordered.
2. an ε-TOD : an ε-transitive set a satisfying ε-trichotomy.
3. an ε-ordinal: an ε-transitive set of ε-transitive sets

The first two are equivalent and stronger. However ε-ordinal implies $\in$-ordinal.

Gimel and Resh operators

Gimel function:

$$\gimel x = x \times \Pi$$

Reish function (recursive):

$$\daleth x = \{(\daleth y, \pi);\ y \in x, \pi \in \Pi\}$$

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In non trivial realizability models (i.e. non-forcing models), $\daleth 2$ has more than two ε-elements (4, infinitely many...).

Recursive ordinals

Fontanella Matthews APAL 2024

- $\daleth\alpha$ and $\hat{\alpha}$ are always ε-ordinals

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- if the realizability model is non trivial (i.e. $\daleth 2 \neq 2$), then ε-TODs are bounded by $\daleth(\kappa^+)$ where κ is the size of the realizability algebra (hence they extensionally form a set)

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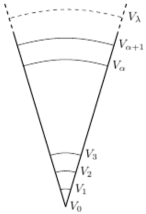
- recursive ordinals are usually not ε-TODs

Large cardinals

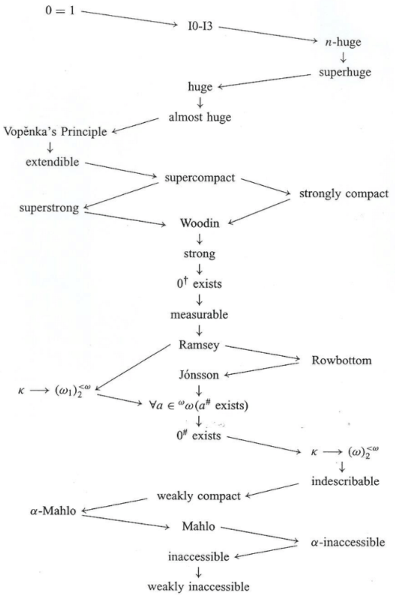
Large cardinals axioms are strong axioms of infinity that assert the existence of uncountable cardinals with various closure properties.

Large cardinals axioms can be ordered by their consistency strength and they all entail the existence of *set*-models of ZF.



.... hence (by Gödel's second incompleteness theorem) the existence of large cardinals cannot be proven within ZF.

The hierarchy of large cardinals



Realizing Large Cardinals

Fontanella Geoffroy Matthews CSL2024

The axioms of inaccessible, Mahlo, measurable and Reinhardt cardinals are preserved by realizability algebras smaller than the cardinals considered.

The work of H. Friedman and A. Ščedrov provided a suitable formulation of large cardinals in intuitionistic set theory.

Realizability models for inaccessible cardinals

An uncountable cardinal κ is strongly inaccessible if it is not a sum of fewer than κ cardinals smaller than κ, and $\gamma < \kappa$ implies $2^\gamma < \kappa$.
The existence of a strongly inaccessible cardinal is equivalent to V_κ being a Grothendieck universe containing ω.

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Let V be a model of ZFC with an inaccessible cardinal κ, and $\mathcal{A} \in V_\kappa$ a realizability algebra. Let $\mathcal{N}$ be the corresponding realizability model, then $\mathcal{N} \Vdash ZF + \text{there exists a Grothendieck universe that contains } \omega.$

idea of the proof

- We realize that $\daleth(V_\kappa)$ is an ε-Grothendieck Universe
- begin an ε-Grothendieck universe implies being a $\in$-Grothendieck universe.

Realizability models for Reinhardt cardinals

A Reinhardt cardinal is the critical point of a non-trivial *elementary embedding* of the universe of sets into itself $j : V \to V$, namely a class function $j : V \to V$ where V is the universe of sets, j is not the identity and for every formula φ and sets $a_1, ..., a_n$ we have

$$\varphi(a_1, .., a_n) \iff \varphi(j(a_1), ..., j(a_n))$$

κ is the critical point means that $j \upharpoonright \kappa = id$ and $j(\kappa) > \kappa$.

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$$\varphi(a_1, .., a_n) \iff \varphi(j(a_1), ..., j(a_n))$$

κ is the critical point means that $j \restriction \kappa = id$ and $j(\kappa) > \kappa$.

The existence of Reinhardt cardinals is inconsistent with ZFC by Kunen's inconsistency theorem, so they are defined only in the context of ZF.

Realizability models for Reinhardt cardinals

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Let κ be a Reinhardt cardinal and $\mathcal{A} \in V_\kappa$ a realizability algebra. Let $\mathcal{N}$ be the corresponding realizability model, then
$\mathcal{N} \Vdash ZF + \text{there exists a Reinhardt cardinal}$

idea of the proof

- Let $j : V \to V$ be an elementary embedding with critical point κ.
- j induces an elementary embedding j^* of $\mathcal{N}$ onto $\mathcal{N}$,
- $j^*(\daleth(\kappa)) \neq \daleth(\kappa)$
- hence j^* is non trivial and $\daleth(\kappa)$ is a Reinhardt cardinal

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Large cardinals in realizability

- these large cardinals axiom are realized from *any* realizability algebra that is “small enough”
- all large cardinals axioms can be realized with this technique but these four large cardinals axioms are “preserved” by small realizability algebras
- realizing large cardinals axioms does not require special terms

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- Realizability is the implementation of this paradigm and originally the scope of realizability was restricted to mathematical intuitionism
- ...but realizability evolved to include classical logic, set theory, the axiom of choice and now even large cardinals

Thank you