Categorical structures for comprehension and context extension

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Introduction

- ’80, Frege introduced comprehension as a basic operation in his Grundlagen (and Russel fixed it).

$$x : X \mid \top \vdash \phi(x) \quad \Longleftrightarrow \quad x \in \{\phi\}$$

- ’70, Lawvere formulated comprehension as a (right) adjoint functor. For this, formulas have to be indexed over contexts of free variables: use Grothendieck fibrations.
- ’80, Ehrhard proposes Grothendieck fibrations for the semantics of type dependency, and formulates context extension as a classifier of terms.

$$x : X \vdash t : A \quad \Longleftrightarrow \quad t : X \to X.A$$

His D-categories also generalise Lawvere’s comprehension: via propositions-as-types, a comprehension classifies the proofs of a formula.

Introduction

- ’90, Categorical frameworks for the semantics of type dependency proliferate: they all form subcategories of Jacob’s comprehension categories.
- Today: when $\{-\} = X.-$ is a right adjoint and what can go on its left. More generally:
  - obtain “free comprehension structures”,
  - even better, try to find settings over which comprehension is monadic (=can be described purely algebraically).

New results are from joint work with Greta Coraglia, Francesco Dagnino, and Andrea Giusto.

Contents

Grothendieck fibrations

Convenient framework to describe structure over a base category of objects.

A fibration is a functor with the possibility of transporting objects between its fibres in a universal way.

$|-| : \mathrm{Pos} \twoheadrightarrow \mathrm{Set}$ is a fibration

$$\begin{tikzcd} (S, f^{-1}[\leq_X]) \arrow[r, "f"'] & X = (|X|, \leq_X) \\ S \arrow[r, "f"'] & |X| \end{tikzcd}$$

$|-| : \mathrm{Grp} \longrightarrow \mathrm{Set}$ is not a fibration

$$\begin{tikzcd} & \langle S \rangle \arrow[dr, "\bar{f}"] & \\ ? \arrow[rr, dashed] & & G \\ & |\langle S \rangle| \arrow[dr, "|\bar{f}|"] & \\ S \arrow[ur, hook] \arrow[rr, "f"'] & & |G| \end{tikzcd}$$

Grothendieck fibrations

Convenient framework to describe structure over a base category of objects.

A fibration is a functor with the possibility of transporting objects between its fibres in a universal way.

$|-|\colon \mathrm{Pos} \twoheadrightarrow \mathrm{Set}$ is a fibration

$$\begin{tikzcd}
Y \arrow[d, dashed] \arrow[dr, bend left, "g"] & \\
(S, f^{-1}[\leq_X]) \arrow[r, "f"'] & X = (|X|, \leq_X) \\
|Y| = S \arrow[r, "f"'] & |X|
\end{tikzcd}$$

$|-|\colon \mathrm{Grp} \longrightarrow \mathrm{Set}$ is not a fibration

$$\begin{tikzcd}
 & \langle S \rangle \arrow[dr, "\bar{f}"] & \\
? \arrow[rr, dashed] & & G \\
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\end{tikzcd}$$

The universal arrow $f\colon (S, f^{-1} \leq_X) \to X$ in Pos is called cartesian.

The syntactic fibration

T a theory in (a fragment of) a first order multi-sorted language.

The syntactic fibration $\mathrm{Syn}_T : \mathcal{C}(T) \longrightarrow\!\!\!\!\rightarrow \mathrm{Ctx}_T$:

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The syntactic fibration $\mathrm{Syn}_T : \mathcal{C}(T) \twoheadrightarrow \mathrm{Ctx}_T$:

$$(y_1 : S'_1, \ldots, y_m : S'_m) \xrightarrow{(t_1,\ldots,t_n)} (x_1 : S_1, \ldots, x_n : S_n)$$

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$$\vec{y} : \vec{S'} \mid \alpha[{}^{t_1}/_{x_1}, \ldots, {}^{t_n}/_{x_n}] \longrightarrow \vec{x} : \vec{S} \mid \alpha$$

$$(y_1 : S'_1, \ldots, y_m : S'_m) \xrightarrow{(t_1,\ldots,t_n)} (x_1 : S_1, \ldots, x_n : S_n)$$

Cartesian arrows in $\mathcal{C}(T)$ are substitutions.

Vertical arrows in $\mathcal{C}(T)$ are logical consequences (g is vertical if $\mathrm{Syn}_T(g) = \mathrm{id}$).

As in every fibration, an arbitrary arrow is given by a vertical one followed by a cartesian one: arrows in $\mathcal{C}(T)$ are logical consequences followed by a substitution.

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The syntactic fibration $\mathrm{Syn}_T : \mathcal{C}(T) \twoheadrightarrow \mathrm{Ctx}_T$:

$$\begin{tikzcd}
\top_{(x_1:\mathbb{N})} \arrow[r] \arrow[d, dashed] & x_1 = x_2 \arrow[d] \\
x_1 + x_1 = 2x_1 \arrow[r] & x_1 + x_2 = 2x_1
\end{tikzcd}$$

$$(x_1 : \mathbb{N}) \xrightarrow{(x_1, x_1)} (x_1 : \mathbb{N}, x_2 : \mathbb{N})$$

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$$(x_1 : \mathbb{N}) \xrightarrow{(x_1, x_1)} (x_1 : \mathbb{N}, x_2 : \mathbb{N})$$

Syn_T is a faithful fibration:

- p is a faithful functor, equivalently
- each fibre $\mathbf{E}_X$ is a poset.

Fibrations from type dependency

From Martin-Löf Type Theory, two fibrations can be constructed.

- Base category: contexts and substitutions.
- Cartesian arrows: substitutions of terms in types
- Vertical arrows: two alternatives
  1. none (obtain a discrete fibration)
  2. given types A, B in context Γ, vertical arrows $A \longrightarrow B$ are terms $\Gamma, A \vdash t : B$ (proof-relevant logical consequences).

More fibrations

$\mathrm{Pred} \twoheadrightarrow \mathrm{Set}$

$$\begin{tikzcd} f^{-1}[S] \subseteq Y \arrow[r, "f"'] & S \subseteq X \\ Y \arrow[r, "f"'] & X \end{tikzcd}$$

$\mathrm{cod} \colon \mathrm{Sub}_{\mathcal{C}} \twoheadrightarrow \mathcal{C}$

$$\begin{tikzcd} Y \times_X A \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, hook] \\ Y \arrow[r, "f"'] & X \end{tikzcd}$$

$\mathrm{Fam}(\mathcal{C}) \twoheadrightarrow \mathrm{Set}$

$$\begin{tikzcd} S' \xrightarrow{kf} \mathrm{Ob}\mathcal{C} \arrow[r] & S \xrightarrow{k} \mathrm{Ob}\mathcal{C} \\ S' \arrow[r, "f"'] & S \end{tikzcd}$$

$\mathrm{cod} \colon \mathcal{C}^2 \twoheadrightarrow \mathcal{C}$

$$\begin{tikzcd} Y \times_X A \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d] \\ Y \arrow[r, "f"'] & X \end{tikzcd}$$

Products in fibrations

$p : \mathcal{E} \twoheadrightarrow \mathcal{B}$ has fibred terminals if it has a rari (right adjoint right inverse):

$$\begin{tikzcd} \mathcal{E} \arrow[d, two heads, "p"', bend right=0] \\ \mathcal{B} \arrow[u, bend right, "\top"'] \end{tikzcd} \quad p \dashv \top \qquad\qquad \begin{tikzcd} A \arrow[d, "\eta_A"] \\ \top_X \\ X \end{tikzcd}$$

Products in fibrations

$p \colon \mathcal{E} \longrightarrow\!\!\!\!\!\rightarrow \mathcal{B}$ has fibred terminals if it has a rari (right adjoint right inverse):

$$\begin{tikzcd} \mathcal{E} \arrow[d, two heads, "p"'] \arrow[d, phantom, shift left=3, "\dashv"] \\ \mathcal{B} \arrow[u, bend right, "\top"'] \end{tikzcd} \qquad \begin{tikzcd} & A \arrow[d, "\eta_A"] \\ \top_Y \arrow[r, "\top_f"] & \top_X \\ Y \arrow[r, "f"] & X \end{tikzcd}$$

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Y \arrow[r, "f"] & X
\end{tikzcd}
$$

$p \colon \mathcal{E} \twoheadrightarrow \mathcal{B}$ has binary fibred products if

$$
\begin{tikzcd}
A \\
A \wedge_X B \arrow[u] \arrow[d] \\
B \\
X
\end{tikzcd}
$$

Products in fibrations

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Y \arrow[r, "f"] & X
\end{tikzcd}$$

$p\colon \mathcal{E} \twoheadrightarrow \mathcal{B}$ has binary fibred products if

$$\begin{tikzcd}
A' \arrow[rr] & & A \\
A' \wedge_Y B' \arrow[u] \arrow[d] \arrow[rr] & & A \wedge_X B \arrow[u] \arrow[d] \\
B' \arrow[rr] & & B \\
Y \arrow[rr, "f"] & & X
\end{tikzcd}$$

Products in fibrations

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$$\begin{tikzcd} & A \arrow[d, "\eta_A"] \\ \top_Y \arrow[r, "\top_f"] & \top_X \\ Y \arrow[r, "f"] & X \end{tikzcd}$$

$p\colon \mathcal{E} \twoheadrightarrow \mathcal{B}$ has binary fibred products if

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If $\mathcal{B}$ has finite products, then $\mathcal{E}$ has finite products and p preserves them.

Contents

Lawvere's comprehension

$p\colon \mathcal{E} \twoheadrightarrow \mathcal{B}$ a bifibration with fibred terminals (and with BCC for all pullbacks).
It has comprehensions if, for every $X \in \mathcal{B}$:

$$\begin{tikzcd} \mathcal{B}/X \arrow[rr, "(Y \xrightarrow{f} X) \mapsto f_! \top_Y"'] \arrow[rr, phantom, shift left=2, "\top"] & & \mathcal{E}_X \arrow[ll, bend right, "\mathrm{L}_X"'] \end{tikzcd}$$

i.e.

$$\begin{tikzcd} \mathcal{B}^2 \arrow[rr, "(-)_! \top_{\mathrm{dom}(-)}"'] \arrow[rr, phantom, shift left=2, "\top"] \arrow[dr, "\mathrm{cod}"'] & & \mathcal{E} \arrow[ll, bend right, "\mathrm{L}"'] \arrow[dl, "p"] \\ & \mathcal{B} & \end{tikzcd}$$

$$\begin{tikzcd} Y \arrow[rr, dashed] \arrow[dr, "f"'] & & \{A\} \arrow[dl, "\mathrm{L}_X A"] \\ & X & \end{tikzcd} \qquad \cong \qquad f_! \top_Y \dashrightarrow A$$

F.W. Lawvere. Equality in hyperdoctrines and the comprehension schema as an adjoint functor. In: A. Heller (Ed.), *Proc. New York Symposium on Application of Categorical Algebra*, AMS, 1970

Lawvere's comprehension

When $\mathcal{B}$ is a regular category (=it has images and finite limits):

$$\begin{tikzcd} \mathcal{B}/X \arrow[r, bend right, "\mathrm{Img}"'] \arrow[r, phantom, "\top"] & \mathrm{Sub}_{\mathcal{B}}(X) \arrow[l, hook'] \end{tikzcd} \quad \text{i.e.} \quad \begin{tikzcd} \mathcal{B}^2 \arrow[rr, bend right, "\mathrm{Img}"'] \arrow[rr, phantom, "\top"] \arrow[dr, "\mathrm{cod}"'] & & \mathrm{Sub}_{\mathcal{B}} \arrow[ll, hook'] \arrow[dl] \\ & \mathcal{B} & \end{tikzcd}$$

$$\begin{tikzcd} Y \arrow[rr, dashed] \arrow[dr, "f"'] & & A \arrow[dl, hook'] \\ & X & \end{tikzcd} \quad \cong \quad \begin{tikzcd} \mathrm{Img}(f) \arrow[rr, dashed] \arrow[dr, hook] & & A \arrow[dl, hook'] \\ & X & \end{tikzcd}$$

Lawvere's comprehension

When $\mathcal{B}$ is a regular category (=it has images and finite limits):

$$\begin{tikzcd} \mathcal{B}/X \arrow[rr, bend right, "\mathrm{Img}"'] \arrow[rr, phantom, "\top"] & & \mathrm{Sub}_{\mathcal{B}}(X) \arrow[ll, hook'] \end{tikzcd} \qquad \text{i.e.} \qquad \begin{tikzcd} \mathcal{B}^2 \arrow[rr, bend right, "\mathrm{Img}"'] \arrow[rr, phantom, "\top"] \arrow[dr, "\mathrm{cod}"'] & & \mathrm{Sub}_{\mathcal{B}} \arrow[ll, hook'] \arrow[dl] \\ & \mathcal{B} & \end{tikzcd}$$

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But $\mathrm{Sub}_{\mathcal{B}}$ has comprehensions even when $\mathcal{B}$ is not regular!

Ehrhard's comprehension

$p \colon \mathcal{E} \twoheadrightarrow \mathcal{B}$ a fibration with fibred terminal objects (i.e. with a rari $\top \vdash p$).

p is a D-category if $\top$ has a right adjoint $\{-\} \colon \mathcal{E} \to \mathcal{B}$.

$$
\begin{tikzcd}
\mathcal{E} \arrow[rr, bend left, "\{-\}"] \arrow[rr, bend right, two heads, "p"'] & & \mathcal{B} \arrow[ll, "\top"'] \arrow[ll, phantom, bend left=20, "\top"] \arrow[ll, phantom, bend right=20, "\top"]
\end{tikzcd}
$$

T. Ehrhard. A categorical semantics of constructions. LICS 1988

$$\top_Y \longrightarrow A \qquad \cong \qquad Y \longrightarrow \{A\}$$

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$$
\begin{tikzcd}
\mathcal{E} \arrow[rr, bend left, "\{-\}"] \arrow[rr, bend right, two heads, "p"'] & \top \\ \top \\ \top & \mathcal{B} \arrow[ll, "\top"']
\end{tikzcd}
$$

T. Ehrhard. A categorical semantics of constructions. LICS 1988

$$
\begin{tikzcd}
\top_Y \arrow[r, "a"] \arrow[d, "t"'] & A \\
f^*A \arrow[ur] & \\
Y \arrow[r, "f=p(a)"] & X
\end{tikzcd}
\qquad \cong \qquad
\begin{tikzcd}
Y \arrow[rr] \arrow[dr, "f"'] & & \{A\} \arrow[dl, "p(\varepsilon_A)"] \\
& X &
\end{tikzcd}
$$

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T. Ehrhard. A categorical semantics of constructions. LICS 1988

terms over f
$y : Y \vdash t : A[f(y)]$

$$\left\{ \begin{tikzcd} \top_Y \arrow[r, "a"] \arrow[d, "t"'] & A \\ f^*A \arrow[ur] & \\ Y \arrow[r, "f = p(a)"] & X \end{tikzcd} \right\} \cong \left\{ \begin{tikzcd} Y \arrow[rr] \arrow[dr, "f"'] & & \{A\} \arrow[dl, "p(\varepsilon_A)"] \\ & X & \end{tikzcd} \right\}$$

substitutions
extending f

Ehrhard's comprehension

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p is a D-category if $\top$ has a right adjoint $\{-\} : \mathcal{E} \to \mathcal{B}$.

$$\begin{tikzcd}
\mathcal{E} \arrow[rr, bend left, "\{-\}"] \arrow[rr, bend right, two heads, "p"'] & \top \atop {\top \atop \top} & \mathcal{B} \arrow[ll, "\top"']
\end{tikzcd}$$

T. Ehrhard. A categorical semantics of constructions. LICS 1988

terms of type A
$x : X \vdash t : A$

$$\left\{ \begin{array}{c} \top_X \xrightarrow{t} A \\ \\ X \xrightarrow{\mathrm{id}} X \end{array} \right\} \cong \left\{ \begin{tikzcd} X \arrow[rr, "s"] \arrow[dr, "\mathrm{id}"'] & & \{A\} \arrow[dl, "p(\varepsilon_A)"] \\ & X & \end{tikzcd} \right\}$$

sections of $p(\varepsilon_A)$

Ehrhard's comprehension

$p\colon \mathcal{E} \twoheadrightarrow \mathcal{B}$ a fibration with fibred terminal objects (i.e. with a rari $\top \vdash p$).

p is a D-category if $\top$ has a right adjoint $\{-\}\colon \mathcal{E} \to \mathcal{B}$.

$$\begin{tikzcd}[column sep=huge]
\mathcal{E} \arrow[r, bend left=40, "\{-\}"{name=U}] \arrow[r, bend right=40, two heads, "p"'{name=D}] & \mathcal{B} \arrow[l, "\top"{name=M, description}]
\arrow[from=U, to=M, phantom, "\top"] \arrow[from=M, to=D, phantom, "\top"]
\end{tikzcd}$$

T. Ehrhard. A categorical semantics of constructions. LICS 1988

Proposition (Jacobs)

A bifibration has Lawvere comprehensions if and only if it is a D-category.

Define $\mathrm{L}\colon \mathcal{E} \to \mathcal{B}^2$ as $\mathrm{L}A := p(\varepsilon_A)\colon \{A\} \to X$.

The simple fibration

$\mathcal{B}$ a category with finite products.

$$\mathsf{s}\mathcal{B} \xrightarrow{\mathsf{s}_{\mathcal{B}}} \mathcal{B}$$

$$\begin{tikzcd}
(X,U) \arrow[d, "(f{,}t)"] & X \arrow[d, "f"'] & X \arrow[d, "f"] & X \times U \arrow[l, "\mathrm{pr}_1"'] \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\langle t{,}f\,\mathrm{pr}_1\rangle"'] \arrow[dr, "t"] & U \\
(Y,V) & Y & Y & Y \times V \arrow[l, "\mathrm{pr}_1"] \arrow[r, "\mathrm{pr}_2"'] & V
\end{tikzcd}$$

The simple fibration

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$$\begin{tikzcd}
\mathrm{s}\mathcal{B} \arrow[rr, two heads, "\mathrm{s}_{\mathcal{B}}"] & & \mathcal{B} \\
(X,U) \arrow[d, "(f,t)"] & & X \arrow[d, "f"'] \\
(Y,V) & & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[d, "f"] & X \times U \arrow[l, "\mathrm{pr}_1"'] \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\langle t, f\mathrm{pr}_1\rangle"'] \arrow[dr, "t"] & U \\
Y & Y \times V \arrow[l, "\mathrm{pr}_1"] \arrow[r, "\mathrm{pr}_2"'] & V
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{s}\mathcal{B} \arrow[rr, "\{-\}"] & & \mathcal{B} \\
(X,U) \arrow[d, "(f,t)"] & & U \times X \arrow[d, "\langle f\mathrm{pr}_1, t\rangle"] \\
(Y,V) & & V \times Y
\end{tikzcd}$$

The simple fibration

$\mathcal{B}$ a category with finite products.

$$\begin{tikzcd}
\mathsf{s}\mathcal{B} \arrow[rr, two heads, "\mathsf{s}_{\mathcal{B}}"] & & \mathcal{B} \\
(X, U) \arrow[d, "{(f,t)}"] & & X \arrow[d, "f"'] \\
(Y, V) & & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[d, "f"] & X \times U \arrow[l, "\mathrm{pr}_1"'] \arrow[r, "\mathrm{pr}_2"] \arrow[d, "{\langle t, f\mathrm{pr}_1\rangle}"'] \arrow[dr, "t"] & U \\
Y & Y \times V \arrow[l, "\mathrm{pr}_1"] \arrow[r, "\mathrm{pr}_2"'] & V
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathsf{s}\mathcal{B} \arrow[rr, "\{-\}"] & & \mathcal{B} \\
(X, U) \arrow[d, "{(f,t)}"] & & U \times X \arrow[d, "{\langle f\mathrm{pr}_1, t\rangle}"] \\
(Y, V) & & V \times Y
\end{tikzcd}$$

Terms $x : X \mid \lambda u.t : U \Rightarrow V$ are classified by the object $V^{X \times U}$ in $\mathcal{B}$, which:

- is not an exponential of V by U in $\mathcal{B}$, but
- is an exponential of V by U in the fibre $\mathsf{s}\mathcal{B}_X$.

The simple fibration

$\mathcal{B}$ a category with finite products.

$$\mathrm{s}\mathcal{B} \xrightarrow{s_{\mathcal{B}}} \mathcal{B} \qquad\qquad\qquad \mathrm{s}\mathcal{B} \xrightarrow{\{-\}} \mathcal{B}$$

$$\begin{tikzcd}
(X,U) \arrow[d, "{(f,t)}"] & X \arrow[d, "f"'] & X \arrow[d, "f"] & X \times U \arrow[l, "\mathrm{pr}_1"'] \arrow[r, "\mathrm{pr}_2"] \arrow[d, "{\langle t, f\mathrm{pr}_1\rangle}"'] \arrow[dr, "t"] & U & (X,U) \arrow[d, "{(f,t)}"] & U \times X \arrow[d, "{\langle f\mathrm{pr}_1, t\rangle}"] \\
(Y,V) & Y & Y & Y \times V \arrow[l, "\mathrm{pr}_1"] \arrow[r, "\mathrm{pr}_2"'] & V & (Y,V) & V \times Y
\end{tikzcd}$$

Fact

$\mathrm{s}\mathcal{B}$ is the Kleisli category of the (fibred) reader comonad

$$\mathcal{B} \times \mathcal{B} \xrightarrow{(X,U) \mapsto (X, X \times U)} \mathcal{B} \times \mathcal{B}$$

The free D-category

$p \colon \mathcal{E} \twoheadrightarrow \mathcal{B}$ a fibration with finite fibred products ($\wedge_X$ binary product in $\mathcal{E}_X$).
$\bigwedge_p$ the comonad on $\mathcal{E} \times_{\mathcal{B}} \mathcal{E}$ such that $\bigwedge_p(A, B) := (A, A \wedge_X B)$. (Not fibred!)

$$
\begin{tikzcd}
\mathcal{E} \times_{\mathcal{B}} \mathcal{E} \arrow[rr, hook, "\mathrm{I}"'] \arrow[rr, phantom, bend left=20, "\perp"] \arrow[dr, "p_1"] & & \mathrm{Kl}(\textstyle\bigwedge_p) \arrow[ll, bend right, "\mathrm{P}"'] \arrow[dl, two heads, "p^{\mathrm{E}}"] \\
& \mathcal{E} \arrow[ul, bend left, "\langle \mathrm{id}, \top \rangle"] \arrow[ul, phantom, "\perp"] &
\end{tikzcd}
\qquad
\begin{tikzcd}
(A, A \wedge_X B) \arrow[d, "{(a, \langle t, a\pi_2 \rangle)}"'] & & (A, B) \arrow[d, "{(a, t)}"] \\
(A', A' \wedge_Y B') & \arrow[l, mapsto, "\mathrm{P}"'] & (A', B')
\end{tikzcd}
$$

Proposition

$p^{\mathrm{E}} := p_1\mathrm{P}$ is a fibration, with rari $\hat{\top} := \mathrm{I}\langle \mathrm{id}, \top \rangle$, and such that $\hat{\top} \dashv \mathrm{pr}_2\mathrm{P}$.

When $p = ! \colon \mathcal{B} \twoheadrightarrow 1$, the comonad $\bigwedge_p$ is the reader comonad and $p^{\mathrm{E}} = s_{\mathcal{B}}$.

The free D-category

Proposition (Dagnino–E.–Giusto)

$(-)^E$ provides a left biadjoint to the forgetful 2-functor

$$\mathbf{FFPDCat} \longrightarrow \mathbf{FFPFib}$$

from the 2-category of D-categories with finite fibred products
to the 2-category of fibrations with finite fibred products.

A. Giusto. Fibrations with comprehensions and their completions. MSc thesis, Università di Genova, 2024

- Clearly, a fibration has at most one structure of Ehrhard comprehension: the induced 2-monad on **FFPFib** should be oplax-idempotent.
- We expect that the biadjunction is monadic: the 2-category of (pseudo) algebras of the monad should be equivalent to **FFPDCat**.

Contents

Comprehensions without unit types

substitutions extending f	$\cong$	terms over f	$\cong$	terms of type $A[f]$	$\cong$	sections of $p(\varepsilon_{A[f]})$

Comprehensions without unit types

$$\begin{matrix}\text{substitutions}\\ \text{extending } f\end{matrix} \cong \left(\begin{matrix}\text{terms}\\ \text{over } f\end{matrix} \cong \begin{matrix}\text{terms}\\ \text{of type } A[f]\end{matrix}\right) \cong \begin{matrix}\text{sections}\\ \text{of } p(\varepsilon_{A[f]})\end{matrix}$$

Comprehensions without unit types

$$\text{substitutions extending } f \;\cong\; \left(\text{terms over } f \;\cong\; \text{terms of type } A[f] \right) \;\cong\; \text{sections of } p(\varepsilon_{A[f]})$$

$$\begin{tikzcd}
Y \arrow[r, "s"] \arrow[rr, bend left, "\bar{f}"] \arrow[dr, bend right, "\mathrm{id}"'] & \{A[f]\} \arrow[r] \arrow[d, "p(\varepsilon_{A[f]})"'] \arrow[dr, phantom, "\lrcorner", very near start] & \{A\} \arrow[d, "p(\varepsilon_A)"] \\
& Y \arrow[r, "f"] & X
\end{tikzcd}$$

Comprehensions without unit types

$$\begin{array}{c}\text{substitutions}\\ \text{extending } f\end{array} \cong \left(\begin{array}{c}\text{terms}\\ \text{over } f\end{array} \cong \begin{array}{c}\text{terms}\\ \text{of type } A[f]\end{array}\right) \cong \begin{array}{c}\text{sections}\\ \text{of } p(\varepsilon_{A[f]})\end{array}$$

$$\begin{tikzcd}
Y \arrow[r, "{(\mathrm{id},t)}"] \arrow[rr, bend left, "{(f,t)}"] \arrow[dr, bend right, "\mathrm{id}"'] & Y.A[f] \arrow[r] \arrow[d, "p(\varepsilon_{A[f]})"'] \arrow[dr, phantom, "\lrcorner", very near start] & X.A \arrow[d, "p(\varepsilon_A)"] \\
& Y \arrow[r, "f"] & X
\end{tikzcd}$$

$$\frac{f : Y \to X \qquad X \vdash A \ \mathtt{type} \qquad Y \vdash t : A[f]}{(f,t) : Y \to X.A}$$

Comprehension categories

A comprehension category is

$$\begin{tikzcd} \mathcal{E} \arrow[rr, "\chi"] \arrow[dr, "p"'] & & \mathcal{B}^2 \arrow[dl, "\mathrm{cod}_{\mathcal{B}}"] \\ & \mathcal{B} & \end{tikzcd}$$

where χ preserves cartesian arrows:

$$\begin{tikzcd} A' \arrow[r] & A & & Y.A' \arrow[r] \arrow[d, "\chi_{A'}"'] \arrow[dr, phantom, "\lrcorner", very near start] & X.A \arrow[d, "\chi_A"] \\ Y \arrow[r, "f"] & X \arrow[rr, phantom, "\longmapsto"] & & Y \arrow[r, "f"] & X \end{tikzcd}$$

(p, χ) is discrete if p is a discrete fibration.
(p, χ) is full if χ is full and faithful.

B. Jacobs. Comprehension categories and the semantics of type dependency. TCS 1993

Comprehension categories - examples

1. Every D-category is a comprehension category (with fibred terminals).

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3. Every full comprehension category with fibred terminals is a D-category:

$$\begin{tikzcd}
\mathcal{E} \arrow[rr, "\chi"] \arrow[dr, bend right, "p"'] \arrow[dr, dashed, bend left] & & \mathcal{B}^2 \arrow[dl, bend right, "\mathrm{cod}"'] \arrow[dl, bend left, "\mathrm{dom}"] \\
& \mathcal{B} \arrow[ul, "\top"] \arrow[ur, "\top"'] &
\end{tikzcd}$$

Comprehension categories - examples

1. Every D-category is a comprehension category (with fibred terminals).
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3. Every full comprehension category with fibred terminals is a D-category.
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Comprehension categories - examples

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4. $\mathsf{DiscFib} \hookrightarrow \mathsf{Cat}^2$ is a full comprehension category with fibred terminals, hence a D-category.
5. $\mathrm{Fam}(\mathcal{C}) \to \mathsf{Set}^2$ is a D-category if $\mathcal{C}$ has a terminal object 1, and it is full if and only if $\mathcal{C}(1, -) \colon \mathcal{C} \to \mathsf{Set}$ is full and faithful. The same holds for the externalisation of an internal category. In particular, the externalisation of PERs in Eff is a D-category.

Comprehension categories - examples

- Models of Martin-Löf Type Theory and Calculus of Constructions can be described either
  - as discrete comprehension categories, or
  - as full (split) comprehension categories, these are (often) D-categories: $\mathrm{IsoFib} \hookrightarrow \mathrm{Gpd}^2$ and $\mathrm{KanFib} \hookrightarrow \mathrm{sSet}^2$.

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For the inclusions (of 2-categories) of

- contextual categories (Cartmell)
- display map categories (Taylor)
- categories with attributes (Cartmell, Moggi, Pitts)
- categories with families (Dybjer)

into comprehension categories, see:

B. Ahrens, P. LeFanu Lumsdaine, P.R. North. Comparing semantic frameworks for dependently-sorted algebraic theories. arXiv:2412.19946 (and talk at EPN-WG6 meeting, Genoa 2025)

Comprehension categories - examples

- Models of Martin-Löf Type Theory and Calculus of Constructions can be described either
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For current research on making sense of arbitrary comprehension categories trying to use vertical arrows to obtain semantics of coercive subtyping, see:

G. Coraglia, J.E. Categorical Models of Subtyping. Post. Proc. TYPES 2023

N. Najmaei, N. van der Weide, B. Ahrens, P. R. North. A Type Theory for Comprehension Categories with Applications to Subtyping. arXiv:2503.10868 (and talk at EPN-WG6 meeting, Genoa 2025)

The free comprehension category

$p \colon \mathcal{E} \twoheadrightarrow \mathcal{B}$ any fibration.

$$\begin{tikzcd}
\mathcal{E}^{\mathsf{J}} \arrow[r, two heads] \arrow[d, two heads, "p^{\mathsf{J}}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{E} \arrow[d, two heads, "p"] \\
\mathrm{ffp}(\mathcal{E}) \arrow[r, two heads, "\mathrm{ffp}(p)"'] & \mathcal{B}
\end{tikzcd}$$

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$$\begin{tikzcd} \mathcal{E}^{\mathrm{J}} \arrow[r] & \mathrm{ffp}(\mathcal{E})^2 \\ (X, \vec{A}, A) \arrow[r, mapsto] & (X, (\vec{A}, A)) \arrow[d, "(\mathrm{id}_X, (\mathrm{id}_{[n]}, \mathrm{id}_{\vec{A}}))"] \\ (X, \vec{A}) & (X, \vec{A}) \end{tikzcd}$$

where $\vec{A} = (A_1, \ldots, A_n)$ with $A_i \in \mathcal{E}_X$, $\quad \mathrm{id}_{\vec{A}} = \mathrm{id}_{A_1}, \ldots, \mathrm{id}_{A_n}$

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$$
\mathcal{E}^{\mathrm{J}} \longrightarrow \mathrm{ffp}(\mathcal{E})^2
$$

$$
\begin{tikzcd}
(Y, \vec{B}, f^*A) \arrow[r, "{(f,(k,\vec{g}),f^{\ulcorner A})}"] & (X, \vec{A}, A) \\
(Y, \vec{B}) \arrow[r, "{(f,(k,\vec{g}))}"'] & (X, \vec{A})
\end{tikzcd}
\longmapsto
\begin{tikzcd}
(Y, (\vec{B}, f^*A)) \arrow[r, "{(f,(k,\vec{g},f^{\ulcorner A}))}"] \arrow[d] & (X, (\vec{A}, A)) \arrow[d, "{(\mathrm{id}_X,(\mathrm{id}_{[n]},\mathrm{id}_{\vec{A}}))}"] \\
(Y, \vec{B}) \arrow[r, "{(f,(k,\vec{g}))}"'] & (X, \vec{A})
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$$

where $\vec{A} = (A_1, \ldots, A_n)$ with $A_i \in \mathcal{E}_X$, $\quad \mathrm{id}_{\vec{A}} = \mathrm{id}_{A_1}, \ldots, \mathrm{id}_{A_n}$,
$k\colon [n] \to [m]$, and $g_i\colon B_{k(i)} \to A_i$ are such that $p(g_i) = f\colon Y \to X$.

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\begin{tikzcd}
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& & \arrow[r, mapsto] & {} & {} \\
(Y, \vec{B}) \arrow[r, "{(f,(k,\vec{g}))}"'] & (X, \vec{A}) & & (Y, \vec{B}) \arrow[r, "{(f,(k,\vec{g}))}"'] & (X, \vec{A})
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$$

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$k\colon [n] \to [m]$, and $g_i\colon B_{k(i)} \to A_i$ are such that $p(g_i) = f\colon Y \to X$.

- p^{J} has fibred terminals if and only if p does.
- the comprehension of p^{J} is not full.

The free comprehension category

Proposition (Dagnino–E.–Giusto)

$(-)^{\mathrm{J}}$ provides a left biadjoint to the forgetful 2-functor

$$\begin{tikzcd} \mathbf{CompCat} \arrow[rr, "\perp"] & & \mathbf{Fib} \arrow[ll, bend right, "(-)^{\mathrm{J}}"'] \end{tikzcd}$$

A. Giusto. Fibrations with comprehensions and their completions. MSc thesis, Università di Genova, 2024

- The induced 2-monad on **Fib** is not oplax idempotent: for p a fibration with finite fibred products, p^{E} has full comprehensions and p^{J} does not.
- Monadic?

Contents

Back to basics

$$\frac{\vdash X\ \mathtt{ctx} \qquad X \vdash A\ \mathtt{type}}{\vdash X.A\ \mathtt{ctx}}$$

context extension

$$\frac{\vdash X\ \mathtt{ctx} \qquad X \vdash A\ \mathtt{type}}{X.A \vdash \mathrm{v}_A : A}$$

assumption

Back to basics

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Categories with families I

A category with families (cwf) consists of

- A Fam-valued presheaf $\mathsf{Tm} \colon \mathcal{C}^{\mathrm{op}} \to \mathsf{Fam}$ where $\mathsf{Tm}(\Gamma) = (\mathsf{Tm}(\Gamma, A))_{A \in \mathsf{Ty}(\Gamma)}$,
- for every $A \in \mathsf{Ty}(\Gamma)$, an arrow $\mathsf{p}_A \colon \Gamma.A \to \Gamma$ and an element $\mathsf{v}_A \in \mathsf{Tm}(\Gamma.A, A[\mathsf{p}_A])$ such that, naturally in Γ,

$$\mathsf{Sect}(\mathsf{p}_A) \xrightarrow[\gamma \mapsto \mathsf{v}_A[\gamma]]{\sim} \mathsf{Tm}(\Gamma, A)$$

Peter Dybjer. Internal type theory. TYPES 1995

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Peter Dybjer. Internal type theory. TYPES 1995

This is equivalent to saying that the forgetful morphism of presheaves $\mathsf{Tm} \to \mathsf{Ty}$ over $\mathcal{C}$ is representable.

S. Awodey. Natural models of homotopy type theory. MSCS 28, 2018

Categories with families II

Apply the Grothendieck construction to the representable $\mathtt{Tm} \longrightarrow \mathtt{Ty}$ to obtain:

$$\begin{tikzcd} \mathtt{Tm} \arrow[rr, bend right, "\Sigma"'] \arrow[dr, "\dot{u}"'] & & \mathtt{Ty} \arrow[ll, bend right, "\Delta"'] \arrow[dl, "u"] \\ & \mathcal{C} & \end{tikzcd}$$

- u and $\dot{u}$ are discrete fibrations.
- Σ is a morphism of discrete fibrations, i.e. $u \circ \Sigma = \dot{u}$.
  We write $\Gamma \vdash t : A$, for $t \in \mathtt{Tm}$, $\Gamma = \dot{u}(t)$, and $A = \Sigma(t)$.
- $\Sigma \dashv \Delta$.

Δ is not a morphism of fibrations: it changes the underlying context!

$$\Delta(\Gamma \vdash A) = \Gamma.A \vdash \mathrm{v}_A : A \quad \text{and} \quad \mathrm{p}_A = u(\varepsilon_A) \colon \Gamma.A = \dot{u}\Delta A \longrightarrow uA = \Gamma.$$

T. Uemura. A general framework for the semantics of type theory. MSCS 33, 2023

Categories with families as discrete comprehension categories

Theorem (Hofmann)

CwFs are equivalent to discrete CompCats.

We can forget the (discrete) fibration of terms $\dot{u}\colon \mathtt{Tm} \to \mathcal{C}$ because, by definition, we can recover it as the fibration of sections:

$$\mathtt{Tm}(\Gamma, A) \xrightarrow[t \mapsto \dot{u}(\eta_t)]{\sim} \mathsf{Sect}(\mathsf{p}_A)$$

M. Hofmann. Syntax and semantics of dependent types. 1997

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Since $\varepsilon_A\colon \Sigma\Delta A \to A$ is cartesian over $\mathrm{p}_A = u(\varepsilon_A)$,

$$\mathtt{Tm}(\Gamma, A) \xrightarrow[t \mapsto \Sigma(\eta_t)]{\sim} \mathsf{Sect}(\varepsilon_A)$$

In fact, terms are coalgebras of the comonad $(\Sigma\Delta, \varepsilon, \Sigma\eta\Delta)$ on $\mathtt{Ty}$.

Weakening-and-contraction comonads

A weakening-and-contraction comonad (w-comonad) consists of a fibration $u\colon \mathcal{E} \to \mathcal{C}$ and a comonad (K, ε, ν) on $\mathcal{E}$ such that

1. for every $A \in \mathcal{E}$, the component $\varepsilon_A\colon KA \to A$ of the counit is cartesian.
2. for every cartesian arrow $f\colon A \to B$ in $\mathcal{E}$ the image in $\mathcal{B}$ under u of the naturality square of ε is a pullback square in $\mathcal{C}$.

$$\begin{tikzcd} uKA \arrow[r, "uK(f)"] \arrow[d, "u(\varepsilon_A)"'] \arrow[dr, phantom, "\lrcorner", very near start] & uKB \arrow[d, "u(\varepsilon_B)"] \\ uA \arrow[r, "u(f)"] & uB \end{tikzcd}$$

B. Jacobs. Categorical Logic and Type Theory. 1999

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uA \arrow[r, "u(f)"] & uB
\end{tikzcd}$$

$$\mathbf{WCmd} \equiv \mathbf{CompCat}$$

$$(u, K, \varepsilon) \longmapsto (u, u(\varepsilon_{(-)}))$$

$$(u, W, \omega) \longleftarrow\!\!\shortmid (u, \chi)$$

where $\omega_A\colon WA \to A$ is a chosen reindexing of A over $\chi_A\colon \Gamma.A \to \Gamma$.

B. Jacobs. Categorical Logic and Type Theory. 1999

The (co-)structure-semantics 2-adjunction

$$\begin{tikzcd}
\mathbf{Cmd} \arrow[rr, hook, shift right, "\mathrm{EM}"'] \arrow[dr, bend right, "\mathrm{Und}"'] \arrow[rr, phantom, "\perp"] & & \mathbf{coTractFun} \arrow[ll, shift right, "\mathrm{C}"'] \arrow[dl, hook, bend left] \\
& \mathbf{Cat} &
\end{tikzcd}$$

A functor $F\colon \mathcal{A} \to \mathcal{C}$ is co-tractable if it has:

- a left Kan extension along itself:

  E. Dubuc. Kan Extensions in Enriched Category Theory. LNM 145, 1970

- an Und-cocartesian lift at the identity comonad on $\mathcal{A}$:

  R. Street. The formal theory of monads. JPAA 2, 1972

The (co-)structure-semantics 2-adjunction

counit $\boldsymbol{\varepsilon}$ is
$\mathrm{id} \colon \mathrm{C} \circ \mathrm{EM} = \mathrm{Id}$

$$
\begin{tikzcd}
\mathbf{Cmd} \arrow[rr, shift right, "\mathrm{EM}"'] \arrow[rr, phantom, "\perp"] \arrow[d, "\mathrm{Id}"'] & & \mathbf{LAdj} \arrow[ll, shift right, "\mathrm{C}"'] \arrow[d, hook] \\
\mathbf{Cmd} \arrow[rr, shift right, "\mathrm{EM}"'] \arrow[rr, phantom, "\perp"] \arrow[dr, bend right, "\mathrm{Und}"'] & & \mathbf{coTractFun} \arrow[ll, shift right, "\mathrm{C}"'] \arrow[dl, bend left] \\
& \mathbf{Cat} &
\end{tikzcd}
$$

unit $\boldsymbol{\eta}_{L,R}$ is
$\mathrm{K}_{L,R} \colon \mathcal{A} \to \mathrm{CoAlg}(LR)$

A functor $F \colon \mathcal{A} \to \mathcal{C}$ is co-tractable if it has:

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  R. Street. The formal theory of monads. JPAA 2, 1972

Generalised categories with families

A generalised category with families (a gcwf for short) is:

$$
\begin{tikzcd}
\dot{\mathcal{E}} \arrow[rr, bend right, "\Sigma"'] \arrow[dr, "\dot{u}"'] & & \mathcal{E} \arrow[ll, bend right, "\Delta"'] \arrow[dl, "u"] \\
& \mathcal{C} &
\end{tikzcd}
$$

$$\dot{\mathcal{E}} \underset{\Sigma}{\overset{\Delta}{\rightleftarrows}} \mathcal{E} \quad (\top)$$

1. u and $\dot{u}$ are Grothendieck fibrations (not necessarily discrete!),
2. Σ is a morphism of fibrations, i.e. $u \circ \Sigma = \dot{u}$ and Σ preserves cartesian arrows.
3. $\Sigma \dashv \Delta$, the unit is $\dot{u}$-cartesian, and the counit is u-cartesian.

G. Coraglia, I. Di Liberti. Context, judgement, deduction. Proc. CatMI 2023

The (co-)structure-semantics adjunction revisited

The structure-semantics 2-adjunction extends to a biadjunction involving pseudo-squares as 1-cells of adjunctions.

$$
\begin{tikzcd}
\mathbf{Cmd} \arrow[d, "\mathrm{Id}"'] \arrow[rrr, hook, shift right=1ex, "\mathrm{EM}"'] \arrow[rrr, phantom, "\perp"] & & & \mathbf{LAdj}^{\cong} \arrow[lll, shift right=1ex, "\mathrm{C}^{\cong}"'] \arrow[d, hook] & \\
\mathbf{Cmd} \arrow[rrr, hook, shift right=1ex, "\mathrm{EM}"'] \arrow[rrr, phantom, "\perp"] \arrow[drr, hook, "\mathrm{EM}"'] \arrow[ddr, bend right, "\mathrm{Und}"'] & & & \mathbf{coTractFun}^{\cong} \arrow[lll, shift right=1ex, "\mathrm{C}^{\cong}"'] \arrow[dr, hook] & \\
 & & \mathbf{Cat}^2 \arrow[rr, hook] \arrow[dl, "\mathrm{cod}"] & & \mathbf{Cat}^{2\cong} \arrow[dlll, bend left, "\mathrm{cod}"] \\
 & \mathbf{Cat} & & &
\end{tikzcd}
$$

The biequivalence

Theorem (Coraglia–E.)

The co-structure-semantics biadjunction between **Cmd** and $\mathbf{LAdj}^{\cong}$ lifts to a biequivalence between **WCmd** and $\mathbf{GCwF}^{\cong}$.

G. Coraglia, J.E. A 2-categorical analysis of context comprehension. TAC 41, 2024

$$\begin{tikzcd}
\mathbf{CompCat} \arrow[r, shift right] \arrow[r, phantom, "\equiv"] & \mathbf{WCmd} \arrow[l, shift right] \arrow[r, shift right, "\hat{\mathrm{EM}}"'] \arrow[r, phantom, "\equiv"] & \mathbf{GCwF}^{\cong} \arrow[l, shift right, "\hat{\mathrm{C}}^{\cong}"'] \\
& \mathbf{WCmd} \arrow[u, "\mathrm{Id}"] \arrow[r, shift right, "\hat{\mathrm{EM}}"'] \arrow[r, phantom, "\perp"] & \mathbf{GCwF} \arrow[l, shift right, "\hat{\mathrm{C}}"'] \arrow[u, hook]
\end{tikzcd}$$

We recover $\mathbf{DiscCompCat} \equiv \mathbf{CwF}$ since the components of pseudo-squares are vertical arrows in a discrete fibration, hence must be identities.

Wrapping up

Comprehension/context extension can be represented in different ways, depending on the available structure (given substitution in the form of a fibration).

- Hyperdoctrines (Lawvere): comprehension of A is the representing arrow of the presheaf $\mathcal{E}((-)_! \top_{\mathrm{dom}(-)}, A)$ on $\mathcal{B}^2$ of images in A.
- D-categories (Ehrhard): comprehension of A is the representing context $X.A$ of the presheaf $\mathcal{E}(\top_{-}, A)$ on $\mathcal{B}$ of global proof-terms of A.
- Comprehension categories (Jacobs): not universal? No structure required.
- Categories with families (Dybjer): comprehension of A is the context of the representing term ΔA of the presheaf $\mathcal{E}(\Sigma(-), A)$ on $\dot{\mathcal{E}}$ of type morphisms into A.

Thank you!