Yeo's Theorem for Locally Colored Graphs: the Path to Sequentialization in Linear Logic

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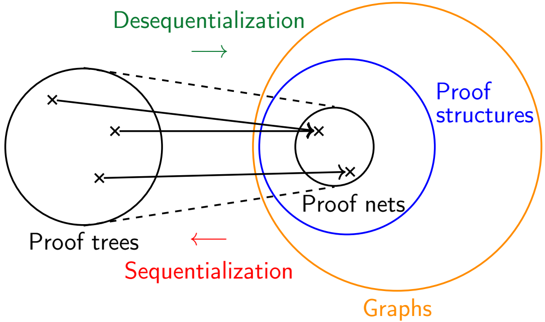
Rome, 12 May 2025

Introduction

Proof nets: graphical, more canonical representation of LL proofs



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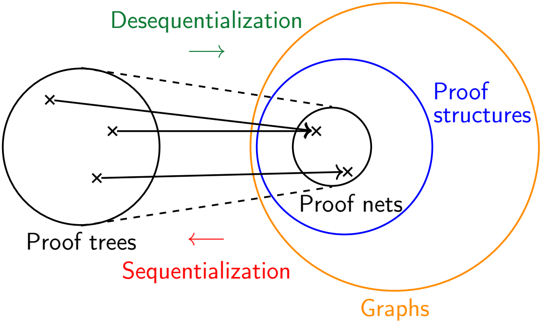

In (unit-free) MLL:
multiple **correctness criteria**,
proofs of sequentialization

Still sequentialization is not considered easy.

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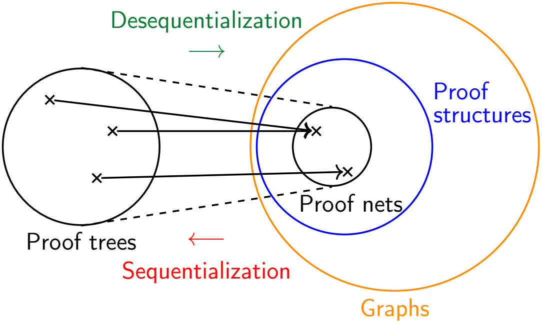

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This talk: easy proof(s) of sequentialization by **splitting** vertices, from a general theorem of **graph theory**
$\longrightarrow$ follows a line of work from Rétoré [Ret03] and Nguyễn [Ngu20]

Outline

- **Multiplicative Linear Logic & Sequentialization**
  - Sequent Calculus & Proof Nets
  - Sequentialization by splitting vertices
- **Simple proof of (a generalized) Yeo's theorem**

Unit-free Multiplicative Linear Logic with Mix

Formulas

$$A ::= X \mid X^{\perp} \mid A \otimes A \mid A \parr A$$

Orthogonality

$$(X^{\perp})^{\perp} = X \qquad (A \otimes B)^{\perp} = A^{\perp} \parr B^{\perp} \qquad (A \parr B)^{\perp} = A^{\perp} \otimes B^{\perp}$$

Rules

$$\frac{}{\vdash A^{\perp}, A}\ (ax) \qquad \frac{\vdash A, \Gamma \quad \vdash B, \Delta}{\vdash A \otimes B, \Gamma, \Delta}\ (\otimes) \qquad \frac{\vdash A, B, \Gamma}{\vdash A \parr B, \Gamma}\ (\parr)$$

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Example of proof structure by desequentialization

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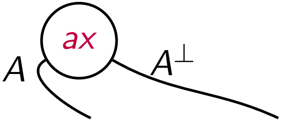

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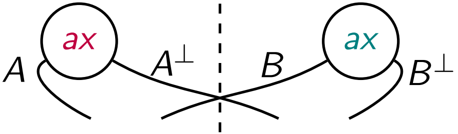

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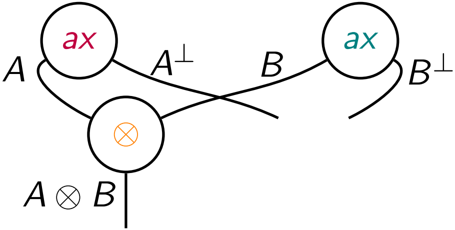

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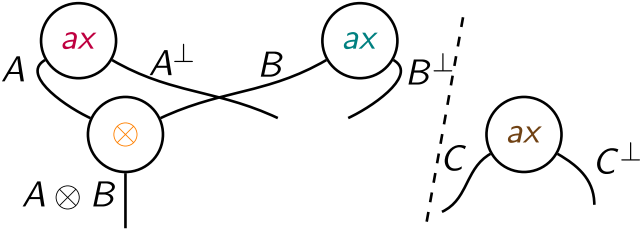

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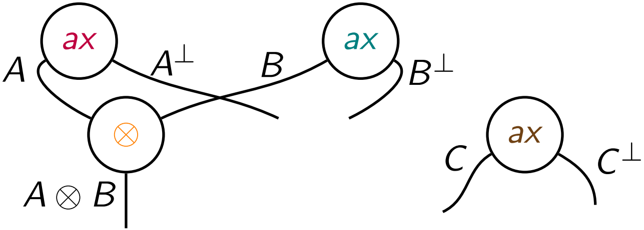

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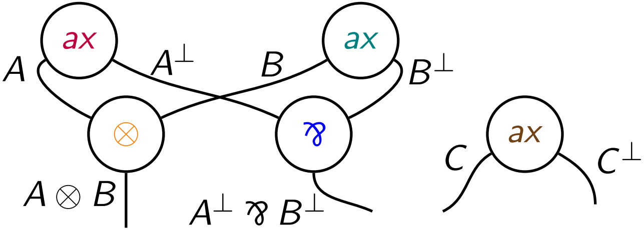

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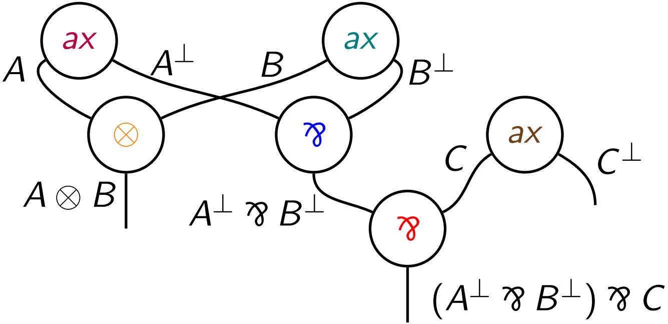

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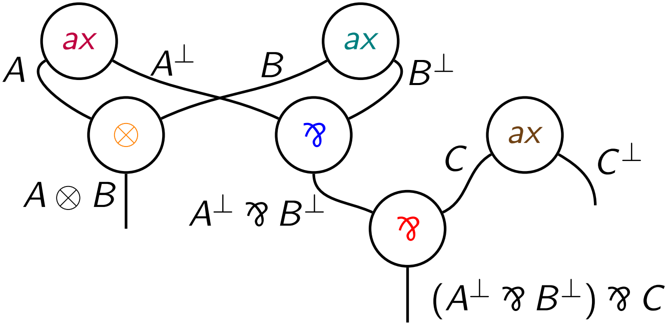
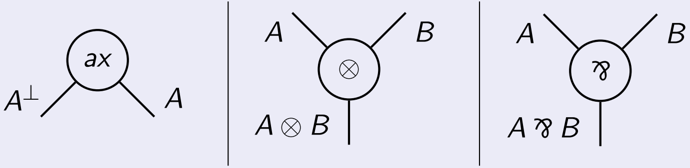


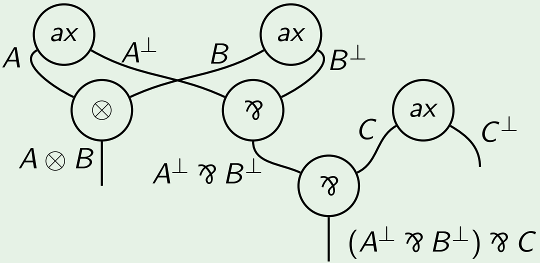
Proof structure

Definition

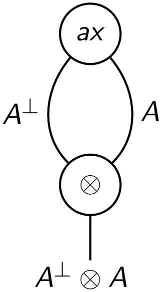
Partial multigraph with labels on vertices → *ax* / $\otimes$ / $⅋$

on edges → formula





Correctness



Correctness

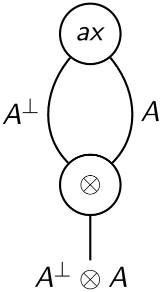
Danos-Regnier Correctness Criterion

Cusp: a ⅋ and its two premises

Switching path/cycle: does not contain any cusp

A proof structure is *correct* if it does not contain any switching cycle

= if every cycle has a cusp



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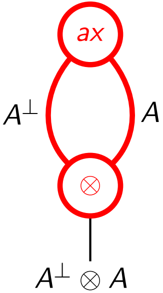

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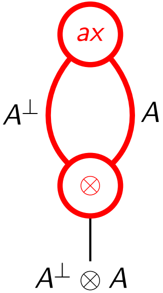

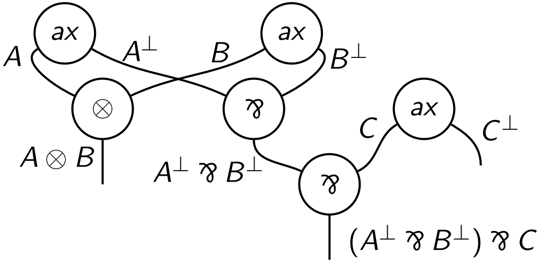

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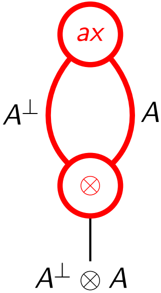

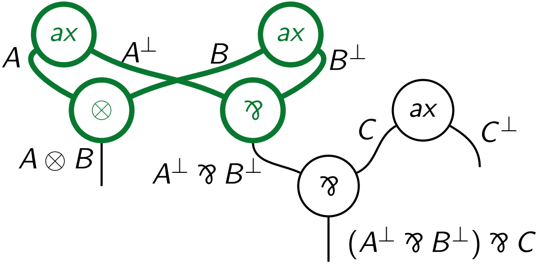

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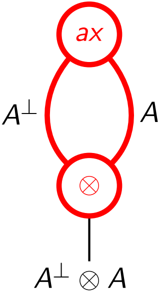
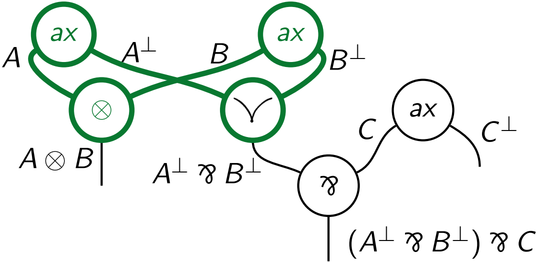




Destination Sequentialization

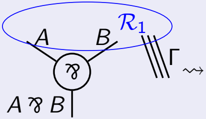
Sequentialization

Given a correct proof structure, there is a proof desequentializing to it.

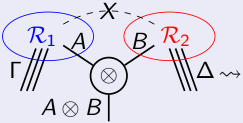
How to prove it? One usual way: by finding a **splitting** vertex

Splitting terminal [Gir87]

- $⅋$ no vertex below
- $\otimes$ no vertex below & not in a cycle



$$\dfrac{\overset{\pi_1}{\vdash A, B, \Gamma}}{\vdash A ⅋ B, \Gamma}\ (⅋)$$



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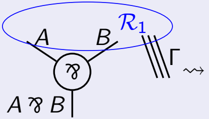

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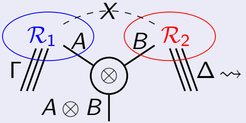

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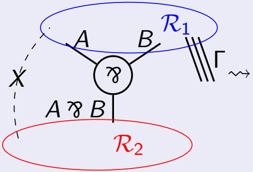
Splitting $\parr$ (aka section) [DR89]

its conclusion edge is not in a cycle



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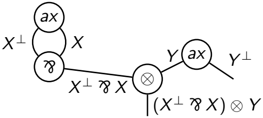
Sequentialization & Yeo's Theorem

Sequentialization

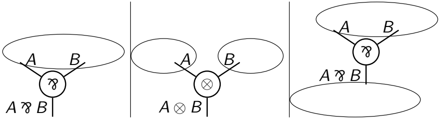
Proof nets



Cusp: a ⅋ and its two premises

no switching (= cusp-free) cycle

$\implies$ $\exists$ splitting vertex



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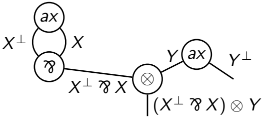

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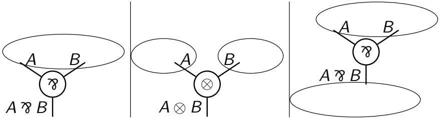

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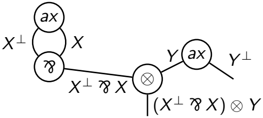

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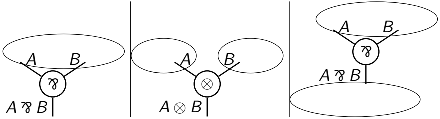
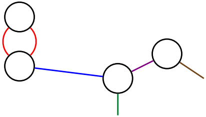


Yeo's Theorem

Edge-colored graphs



Cusp: a vertex and two of its edges of the same color

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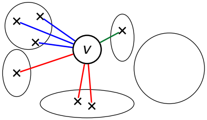

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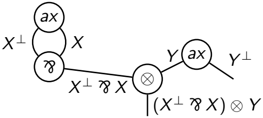

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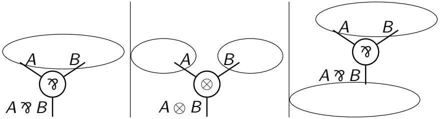

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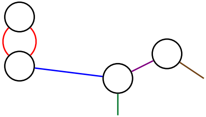

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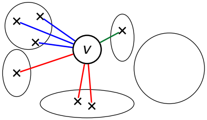


Encoding

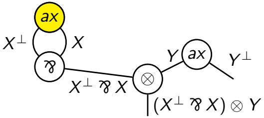
premises of a ⅋ = same color

all other edges of different colors

Sequentialization & Yeo's Theorem

Sequentialization

Proof nets



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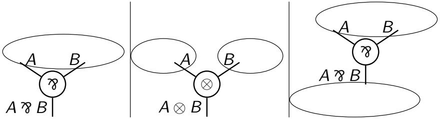

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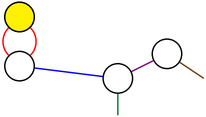

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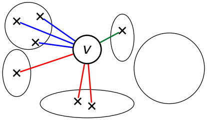

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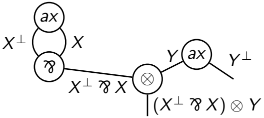
Sequentialization

Proof nets

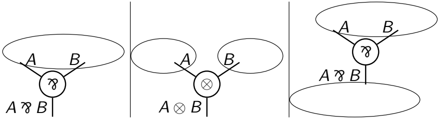


Cusp: a ⅋ and its two premises

no switching (= cusp-free) cycle

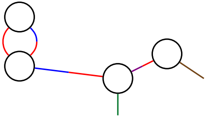
$\Longrightarrow$ $\exists$ splitting vertex

= is a cusp of all its cycles



Generalized Yeo's Theorem

Half-Edge-colored graphs



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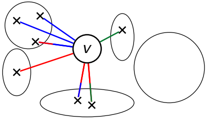


Encoding

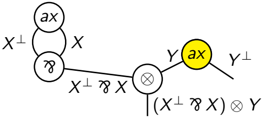
premises of a ⅋ = same color

all other edges of different colors

Sequentialization & Yeo's Theorem

Sequentialization

Proof nets



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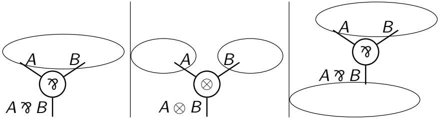

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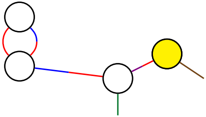

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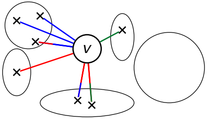

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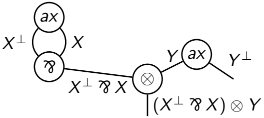

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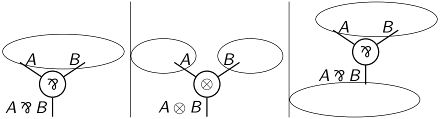

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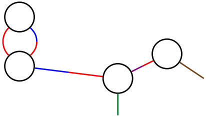

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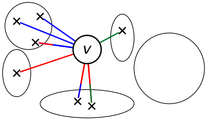

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Outline

- Multiplicative Linear Logic & Sequentialization
  - Sequent Calculus & Proof Nets
  - Sequentialization by splitting vertices
- **Simple proof of (a generalized) Yeo's theorem**

Strict Partial Order on Vertices

Main idea: follow a path evidence of progression = a strict partial order $\lhd$

Goal: a $\lhd$-maximal vertex is splitting

Definition

$v \quad \lhd \quad u$ means there is a path p such that:

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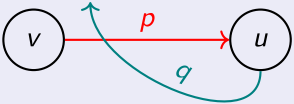
Proof: $\lhd$ is a strict partial order.

Irreflexivity: by definition.

Transitivity: if $(v, \alpha) \overset{p}{\lhd} (u, \beta) \overset{q}{\lhd} (w, \gamma)$ then $(v, \alpha) \overset{p \cdot q}{\lhd} (w, \gamma)$.

(1) ?

(2) ?



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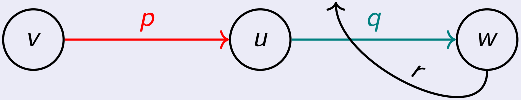

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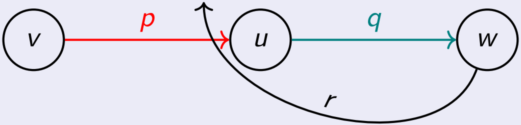

Key (and sole) intermediate lemma

Cusp Minimization

Let ω be a cycle with a cusp at u of color β, but no cusp at v. If there is a simple open cusp-free path q starting from u with color not β and going back on ω, then either there exists a cusp-free cycle or there is a cycle ω' with no cusp at v and strictly less cusps than ω.

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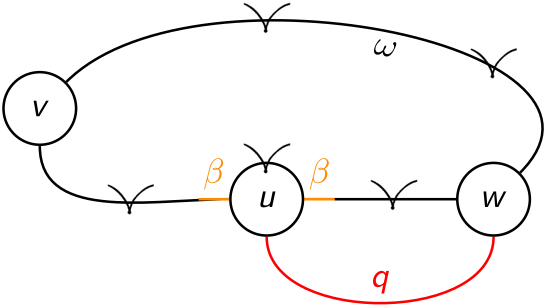

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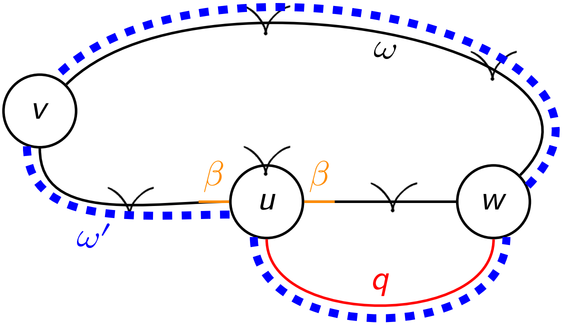

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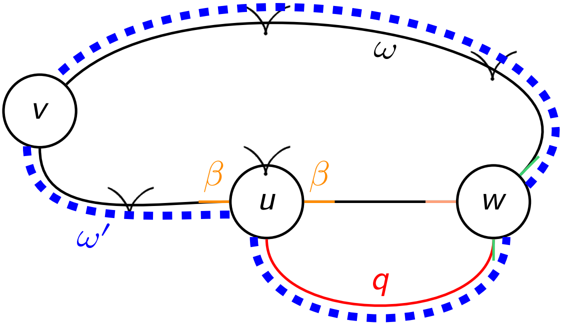

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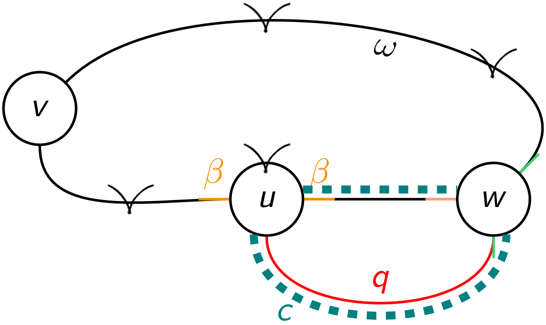


$\lhd$-maximal is splitting

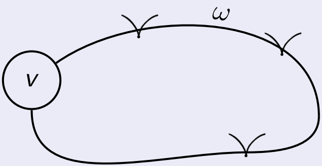
Lemma

Assume v is not splitting. For any color α, there exists (u, β) such that $(v, \alpha) \lhd (u, \beta)$. Furthermore, there is a cusp at u of color β.

Proof.

v not splitting $\implies$ cycle ω with no cusp at v

- *w.l.o.g.* starting color of ω is not α
- *w.l.o.g.* ω has a minimal number of cusps



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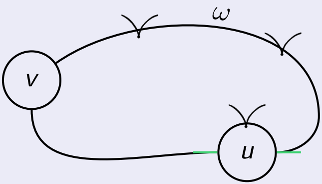

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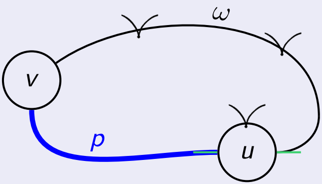

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$(v, \alpha) \overset{p}{\lhd} (u, \beta)$?

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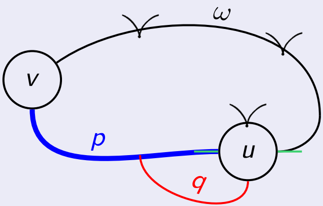
$(v, \alpha) \overset{p}{\lhd} (u, \beta)$? Yes, by Cusp Minimization.



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Generalized Yeo's Theorem

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In a graph G with an half-edge coloring, pose P a set of vertex-color pairs containing at least all (v, α) such that there is a cusp at v with half-edges of color α. If G has no cusp-free cycle, the vertex of any $\lhd$-maximal element of P is splitting.

Proof.

A non-splitting vertex is smaller than some vertex in P. □

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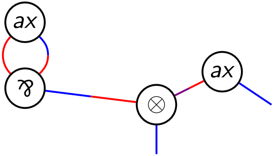
Back to (colored) proof nets: cusp = 

We get a vertex:



Splitting with P all vertex-color pairs

Splitting $\parr$ or $\otimes$ with P all $\parr$- and $\otimes$-color pairs

Splitting $\parr$ with P all $\parr$-color pairs

Splitting terminal with $P := \{(v, \alpha) \mid$
v is a $\parr$ or $\otimes$ and α is the color of one of its premises$\}$

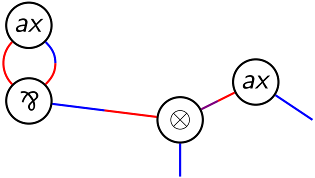
Conclusion

Sequentialization

Given a correct proof structure, there is a proof desequentializing to it.

- Sequentialization by splitting vertices from Yeo's theorem *by only defining a coloring*
- No other encoding $\rightarrow$ can translate our simple proof of Yeo as one of sequentialization (*i.e.* just redefine what a cusp is)
- Other theorems in graph theory, known to be equivalent to Yeo's theorem, can be proved easily this way – only by defining a coloring
- Can be extended to proof nets with additives [HG05]
- Proof simple enough to be formalized in ROCQ

Thank you!



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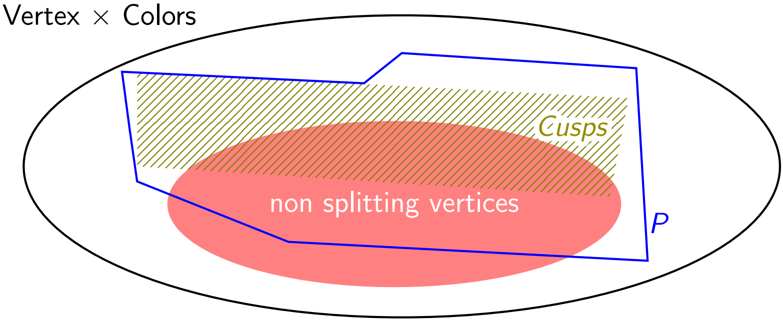
Interest of the parameter P



(maximum elements for $\lhd$ are on top)

Sequentialization and Graph Theory

Sequentialization [HG05]

MALL *Proof nets are exactly the images of proofs.*

Sequentialization and Graph Theory

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↑

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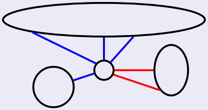

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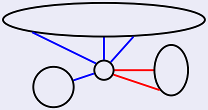
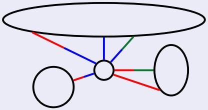


encoding

all

w/o encoding

Yeo with local coloring

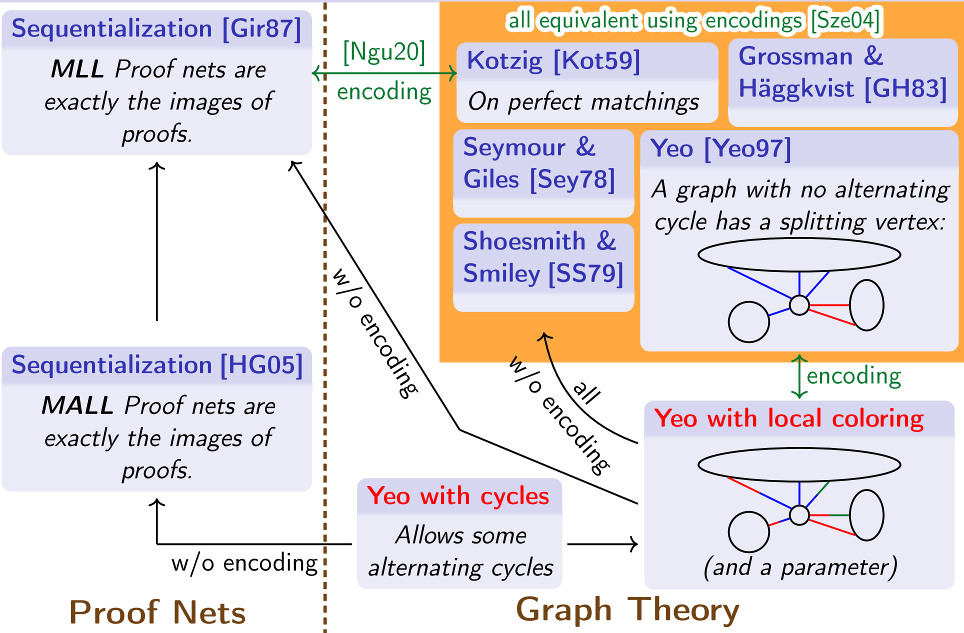


(and a parameter)

Proof Nets

Graph Theory

Sequentialization and Graph Theory



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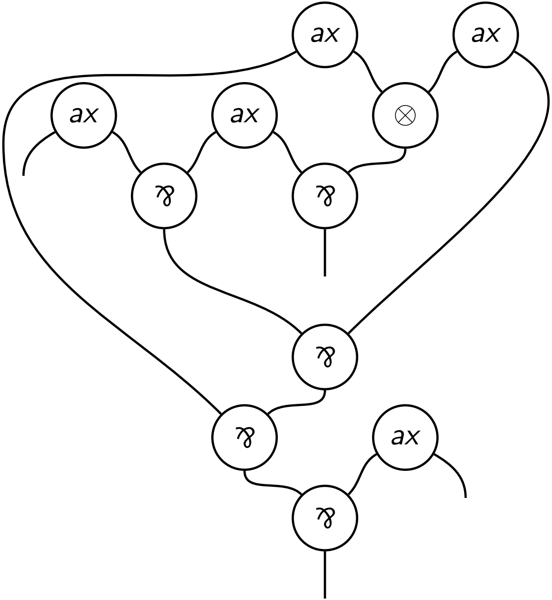

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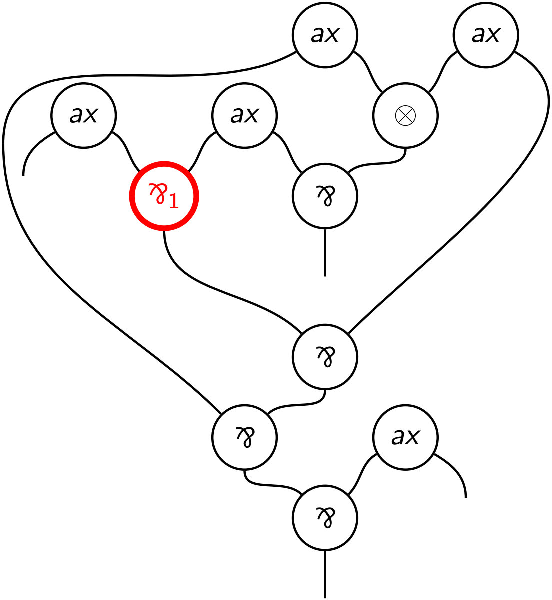

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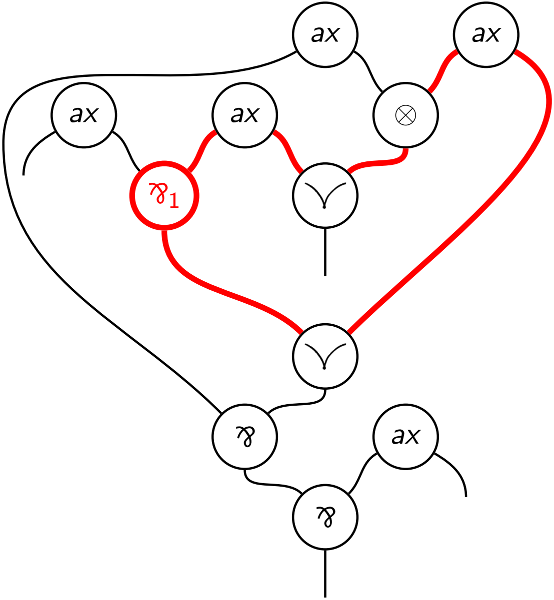
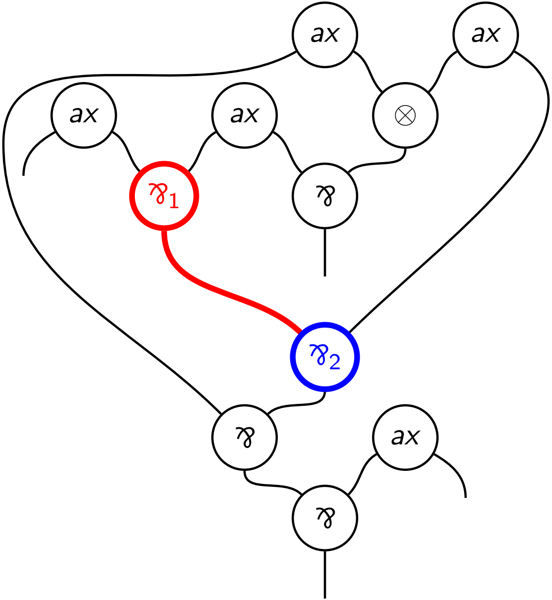
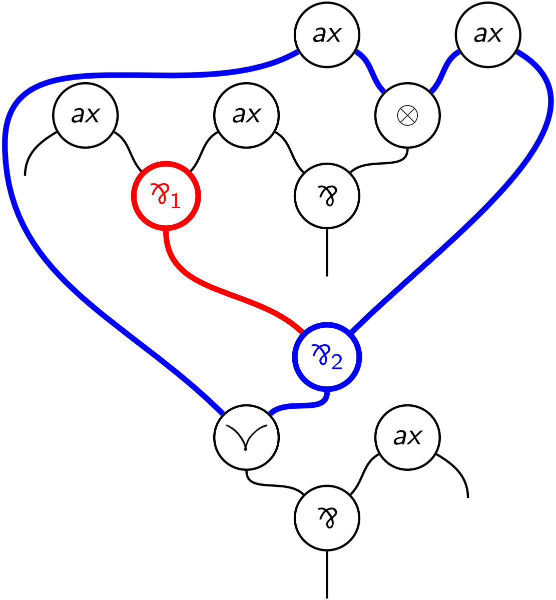


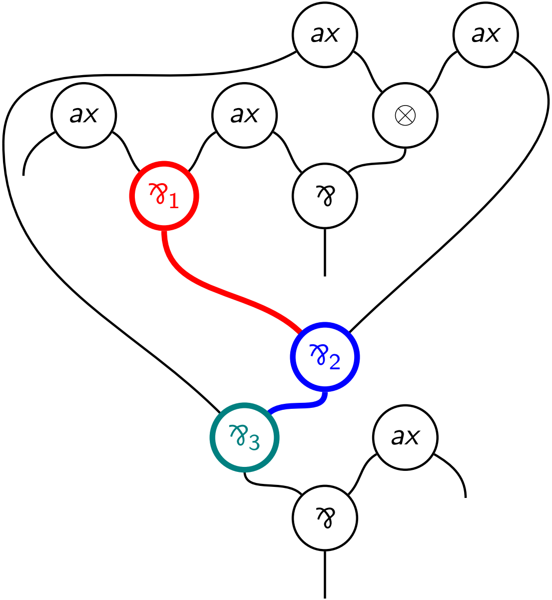
Finding a splitting ⅋ on an example



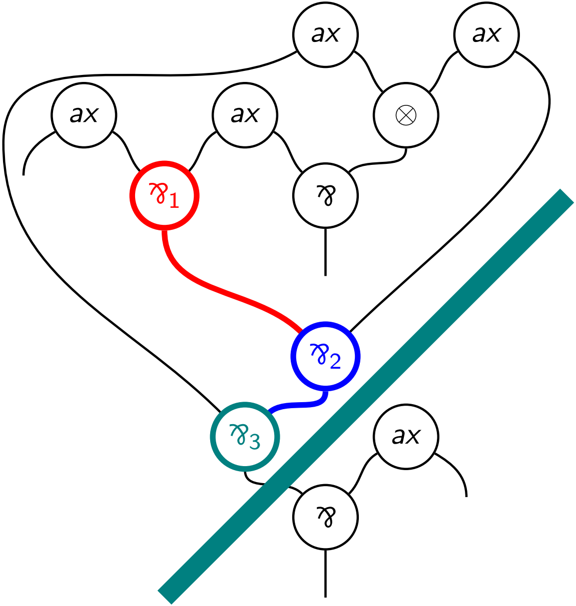
Finding a splitting ⅋ on an example



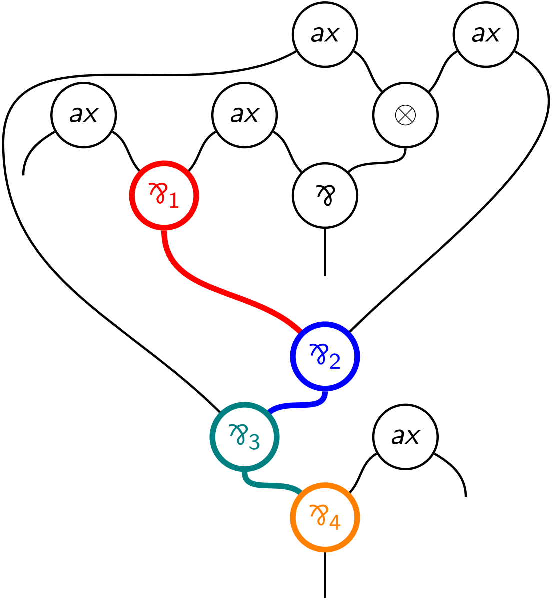
Finding a splitting ⅋ on an example



Finding a splitting ⅋ on an example



Finding a splitting ⅋ on an example



Finding a splitting ⅋ on an example