On the tropical geometry of probabilistic programming languages

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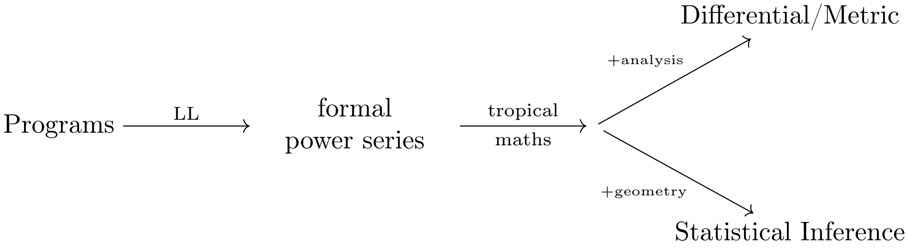
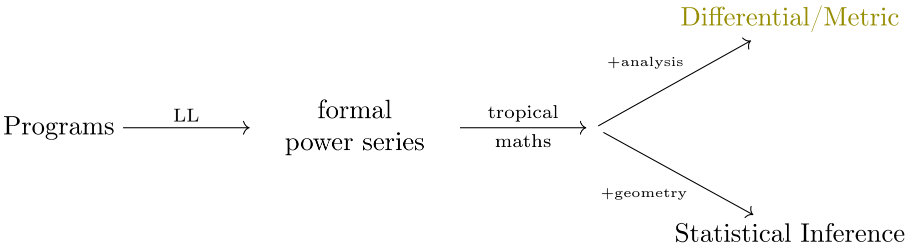
UNIVERSITY OF BATH

IRN $\langle \mathrm{L} \mid \mathrm{I} \rangle$ *Kickoff meeting* *12/05/2025*


Programs
LL
formal
power series
tropical
maths
+analysis
Differential/Metric
+geometry
Statistical Inference



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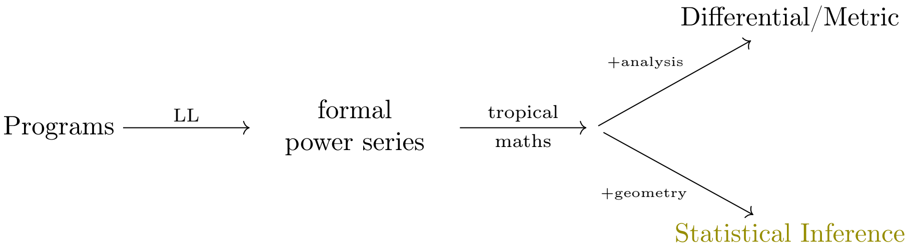
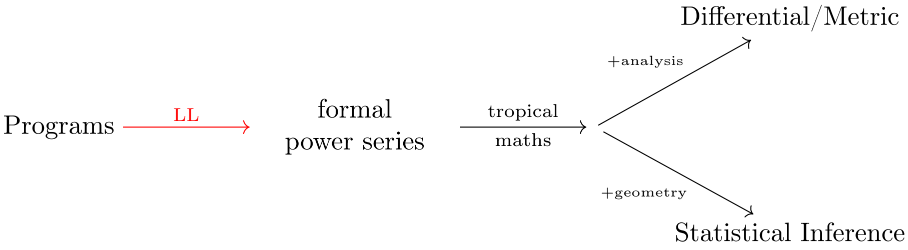
Metric & Differential Analysis of Effectful Programs.



Barbarossa, Pistone – Draft

Tropical Geometry of Probabilistic Programming Languages.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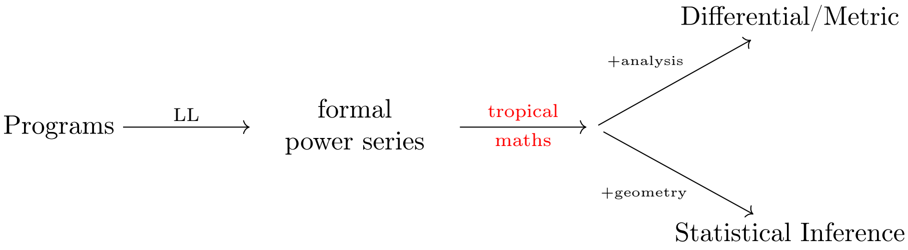
Prog. Lang.	$\longrightarrow$	$Q\mathrm{Rel}_!$
Type	$\longmapsto$	formal variables set
Program	$\longmapsto$	formal power series

Prog. Lang.	$\longrightarrow$	$Q\mathrm{Rel}_!$
Type A	$\longmapsto$	formal variables set $x_{[\![A]\!]} := \{x_a \mid a \in [\![A]\!]\}$
Program $\vec{x} : \vec{A} \vdash M : B$	$\longmapsto$	formal power series $[\![\vec{x} : \vec{A} \vdash M : B]\!] \in Q\{\!\{x_{[\![A]\!]}\}\!\}^{[\![B]\!]}$

Say the language is probabilistic. Then for $Q = \mathbb{R}_{\geq 0}^{+\infty}$ we have

$$[\![\vdash M : \mathrm{Bool}]\!]_i = \mathbb{P}(M \twoheadrightarrow i)$$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$$\mathbb{T} := \begin{array}{l} \text{semiring } [0, +\infty] \text{ with} \\ \text{add} := \inf, \ \text{zero} := +\infty, \\ \text{multiply} := +, \ \text{one} := 0 \end{array}$$

$$\sum_{n \in \mathbb{N}} a_n x^n \in [0,1]\{\!\{x\}\!\} \ \mapsto \ \inf_{n \in \mathbb{N}} \{nx - \ln a_n\} \in \mathbb{T}\{\!\{x\}\!\}$$

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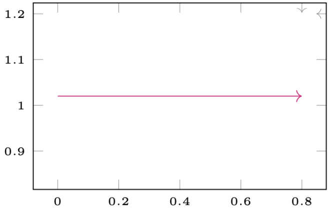
$$\varphi_0(x) = 1$$



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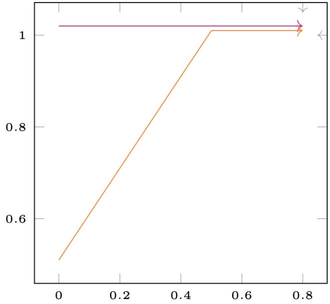
$\varphi_0(x) = 1$

$\varphi_1(x) = \min\{x + \frac{1}{2}, 1\}$



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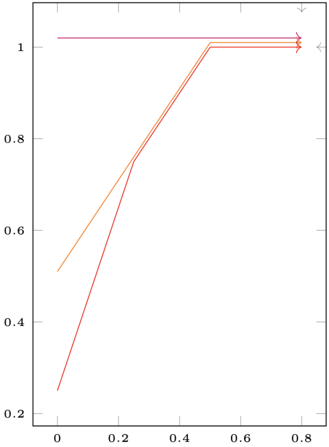

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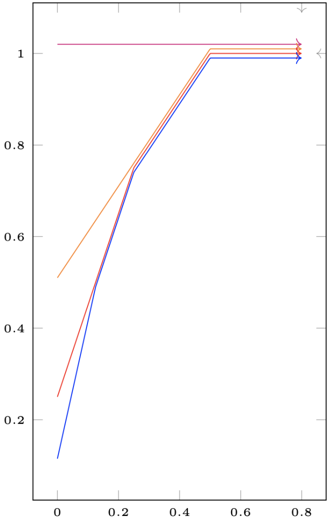

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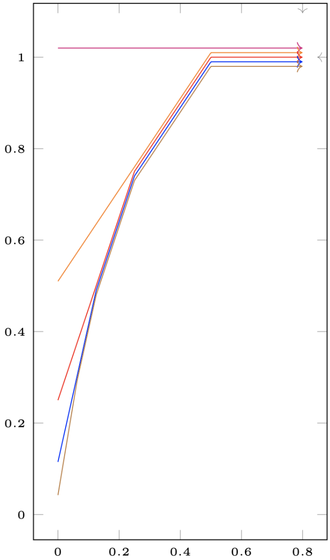
$$\varphi_4(x) = \min\{4x + \tfrac{1}{16}, 3x + \tfrac{1}{8}, 2x + \tfrac{1}{4}, x + \tfrac{1}{2}, 1\}$$



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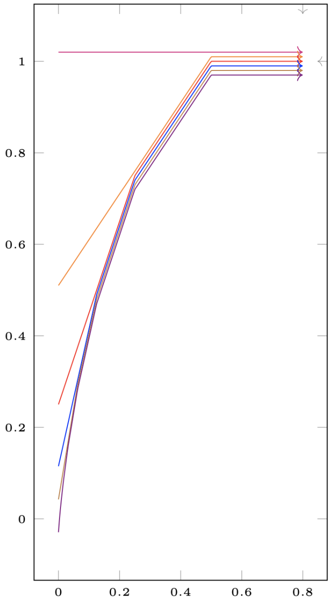
$$\vdots$$

$$\varphi(x) = \inf_n \left\{nx + \tfrac{1}{2^n}\right\}$$



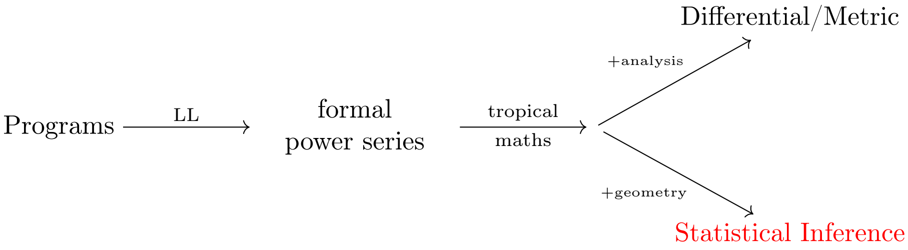
Intractable problems (e.g. root finding, optimization)

Intractable problems (e.g. root finding, optimization)

$\downarrow$ tropicalization:
$+ \mapsto \min$
$\times \mapsto +$

Combinatorial (and sometimes tractable!) ones

- tropical roots are found in linear time
- likelihood estimation in statistical models
- machine learning (ReLU networks)
- optimal routing paths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$$\lambda\text{-}calculus + \frac{\vdash M : A \quad \vdash N : A}{\vdash M \oplus_X N : A} + Arithmetic + Conditionals + \frac{\vdash M : A \to A}{\vdash \boldsymbol{fix}.M : A}$$

$$M \oplus_X N \xrightarrow{X} M$$

$$M \oplus_X N \xrightarrow{\overline{X}} N$$

$$M := \big(\text{True} \oplus_X \text{False}\big) \oplus_X \Big(\big(\text{True} \oplus_X \text{False}\big) \oplus_X \big(\text{False} \oplus_X \text{True}\big)\Big)$$

$$M := \big(\text{True} \oplus_X \text{False}\big) \oplus_X \Big(\big(\text{True} \oplus_X \text{False}\big) \oplus_X \big(\text{False} \oplus_X \text{True}\big)\Big)$$

$$\begin{aligned} P_{ll} &= X^2 \\ P_{rll} &= X^2\overline{X} \\ P_{rrr} &= \overline{X}^3 \end{aligned}$$

$$M := \big(\text{True} \oplus_X \text{False}\big) \oplus_X \Big(\big(\text{True} \oplus_X \text{False}\big) \oplus_X \big(\text{False} \oplus_X \text{True}\big)\Big)$$

$$P_{ll} = X^2$$
$$P_{rll} = X^2\overline{X}$$
$$P_{rrr} = \overline{X}^3$$

What is the most likely path $M \twoheadrightarrow \text{True}$?

$$M := (\text{True} \oplus_X \text{False}) \oplus_X \Big((\text{True} \oplus_X \text{False}) \oplus_X (\text{False} \oplus_X \text{True})\Big)$$

ll	rll	rrr	lr	rlr	rrl
X^2	$X^2\overline{X}$	$\overline{X}^3$	$X\overline{X}$	$X\overline{X}^2$	$X\overline{X}^2$

R
$\downarrow$
O

R	True	False
ll	1	0
rll	1	0
rrr	1	0
lr	0	1
rlr	0	1
rrl	0	1

Hidden Markov Model

Maximum A Posteriori Estimation:
For fixed $X, \overline{X}$, given the observation "$M \twoheadrightarrow$ True", which is its most likely explanation?

$$M := (\text{True} \oplus_X \text{False}) \oplus_X \Big((\text{True} \oplus_X \text{False}) \oplus_X (\text{False} \oplus_X \text{True}) \Big)$$

ll	rll	rrr	lr	rlr	rrl
X^2	$X^2\overline{X}$	$\overline{X}^3$	$X\overline{X}$	$X\overline{X}^2$	$X\overline{X}^2$

$R \downarrow O$

R	True	False
ll	1	0
rll	1	0
rrr	1	0
lr	0	1
rlr	0	1
rrl	0	1

Hidden Markov Model

Maximum A Posteriori Estimation:
For fixed $X, \overline{X}$, given the observation "$M \twoheadrightarrow$ True", which is its most likely explanation?

→ find a reduction ω_0 maximizing $\mathbb{P}(R = \omega_0 \mid O = \text{True})$:

$$\mathbb{P}(R = \omega_0) = \max\{X^2, X^2\overline{X}, \overline{X}^3\}$$

$$M := (\text{True} \oplus_X \text{False}) \oplus_X \Big((\text{True} \oplus_X \text{False}) \oplus_X (\text{False} \oplus_X \text{True}) \Big)$$

ll	rll	rrr	lr	rlr	rrl
X^2	$X^2\overline{X}$	$\overline{X}^3$	$X\overline{X}$	$X\overline{X}^2$	$X\overline{X}^2$

$R \downarrow O$

R	True	False
ll	1	0
rll	1	0
rrr	1	0
lr	0	1
rlr	0	1
rrl	0	1

Hidden Markov Model

Maximum likelihood estimation:
Given the observation "$M \twoheadrightarrow$ True", which value of $X, \overline{X}$ makes the explanation rll as likely as possible?

$$\operatorname*{argmax}_{X,\overline{X}} \frac{X^2\overline{X}}{\mathbb{P}(M \twoheadrightarrow \text{True})}$$

$$M := \mathbf{fix}.(\lambda x. \star \oplus_X x) \twoheadrightarrow \star \oplus_X M$$

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$M \twoheadrightarrow_X \star$	X
$M \twoheadrightarrow_{\overline{X}} M \twoheadrightarrow_X \star$	$X\overline{X}$
$M \twoheadrightarrow_{\overline{X}} M \twoheadrightarrow_{\overline{X}} M \twoheadrightarrow_X \star$	$X\overline{X}^2$
$\dots$	

$$M := \mathbf{fix}.(\lambda x. \star \oplus_X x) \twoheadrightarrow \star \oplus_X M$$

0	1	2	3	...
X	$X\overline{X}$	$X\overline{X}^2$	$X\overline{X}^3$	...

R
$\downarrow$
O

R	$\star$
0	1
1	1
2	1
3	1
$\vdots$	$\vdots$

Hidden Markov Model

Maximum A Posteriori Estimation:
given the observation "M terminates",
which is its most likely explanation?

$$M := \mathbf{fix}.(\lambda x. \star \oplus_X x) \twoheadrightarrow \star \oplus_X M$$

0	1	2	3	...
X	$X\overline{X}$	$X\overline{X}^2$	$X\overline{X}^3$	...

R
$\downarrow$
O

R	$\star$
0	1
1	1
2	1
3	1
$\vdots$	$\vdots$

Hidden Markov Model

Maximum A Posteriori Estimation:
given the observation "M terminates",
which is its most likely explanation?

$$\operatorname*{argmax}_i X\overline{X}^i$$

$$F := \lambda f y.\mathrm{ifte}(Oy,\ \star,\ f(Ny))$$

$$M_i := (\mathbf{fix}.F)(N^i D) \twoheadrightarrow \mathrm{ifte}(O(N^i D),\ \star,\ M_{i+1})$$

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$$\begin{aligned}
\mathbb{P}(R = 1) &= \mathbb{P}(OD = 0) \\
\mathbb{P}(R = i + 1) &= \mathbb{P}(O(N^i D) = 0) \textstyle\prod_{j=0}^{i-1} \mathbb{P}(O(N^j D) \neq 0)
\end{aligned}$$

$$R \downarrow O$$

R	$\star$
0	1
1	1
2	1
3	1
⋮	⋮

Hidden Markov Model

Maximum A Posteriori Estimation:
given the observation "M_0 terminates", which is its most likely explanation?

$$F := \lambda f y.\mathrm{ifte}(Oy,\ \star,\ f(Ny))$$

$$M_i := (\mathbf{fix}.F)(N^i D) \twoheadrightarrow \mathrm{ifte}(O(N^i D),\ \star,\ M_{i+1})$$

$$\begin{array}{lcl} \mathbb{P}(R=1) & = & \mathbb{P}(OD=0) \\ \mathbb{P}(R=i+1) & = & \mathbb{P}(O(N^i D)=0)\prod_{j=0}^{i-1}\mathbb{P}(O(N^j D)\neq 0) \end{array}$$

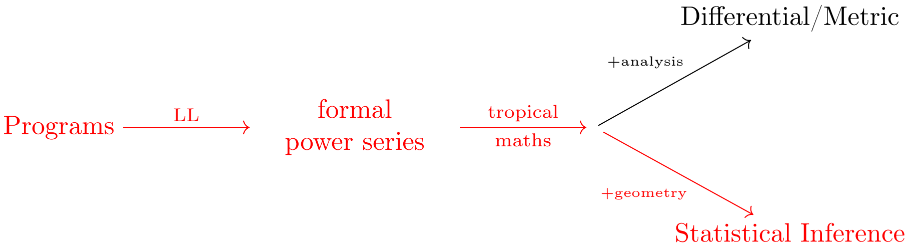
$$\begin{array}{c} R \\ \downarrow \\ O \end{array}$$

R	$\star$
0	1
1	1
2	1
3	1
$\vdots$	$\vdots$

Hidden Markov Model

Maximum A Posteriori Estimation:
given the observation "M_0 terminates", which is its most likely explanation?

$$\underset{i\geq 1}{\mathrm{argmax}}\ \mathbb{P}(R=i)$$



Tropically weighted relational semantics of probabilistic λ-calculi is a natural framework to study statistical inference of Probabilistic Models.

$$M := \mathbf{fix}.(\lambda x. \star \oplus_X x) \twoheadrightarrow \star \oplus_X M$$

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$M \twoheadrightarrow_X \star$	X
$M \twoheadrightarrow_{\overline{X}} M \twoheadrightarrow_X \star$	$X\overline{X}$
$M \twoheadrightarrow_{\overline{X}} M \twoheadrightarrow_{\overline{X}} M \twoheadrightarrow_X \star$	$X\overline{X}^2$
$\dots$	

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$\dots$	

$$\mathbb{P}(M\downarrow) = \sum_{n\in\mathbb{N}} X\overline{X}^n$$

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$M \twoheadrightarrow_{\overline{X}} M \twoheadrightarrow_X \star$	$X\overline{X}$
$M \twoheadrightarrow_{\overline{X}} M \twoheadrightarrow_{\overline{X}} M \twoheadrightarrow_X \star$	$X\overline{X}^2$
$\ldots$	

$$\mathbb{P}(M\downarrow) = \sum_{n \in \mathbb{N}} X\overline{X}^n$$

$$\mathbf{t}\mathbb{P}(M\downarrow) = \inf_{n \in \mathbb{N}}\{X + n\overline{X}\}.$$

$$M := \mathbf{fix}.(\lambda x. \star \oplus_X x) \twoheadrightarrow \star \oplus_X M$$

$$M \twoheadrightarrow_X \star \qquad p$$

$$M \twoheadrightarrow_{\overline{X}} M \twoheadrightarrow_X \star \qquad pq$$

$$M \twoheadrightarrow_{\overline{X}} M \twoheadrightarrow_{\overline{X}} M \twoheadrightarrow_X \star \qquad pq^2$$

$$\dots$$

$$\mathbb{P}(M\downarrow)^{!}(X := p, \overline{X} := q := 1-p) = \sum_{n\in\mathbb{N}} pq^n = \frac{p}{1-q} = 1$$

$$\mathbf{t}\mathbb{P}(M\downarrow) = \inf_{n\in\mathbb{N}}\{X + n\overline{X}\}.$$

$$M := \mathbf{fix}.(\lambda x. \star \oplus_X x) \twoheadrightarrow \star \oplus_X M$$

$M \twoheadrightarrow_X \star$	x
$M \twoheadrightarrow_X M \twoheadrightarrow_X \star$	$x + y$
$M \twoheadrightarrow_X M \twoheadrightarrow_X M \twoheadrightarrow_X \star$	$x + 2y$
$\dots$	

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$M \twoheadrightarrow_X M \twoheadrightarrow_X \star$	$x + y$
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$M \twoheadrightarrow_{\overline{X}} M \twoheadrightarrow_{\overline{X}} M \twoheadrightarrow_X \star$	pq^2
$\dots$	

$$[\![M]\!]^!(p, 1-p) = \mathbb{P}(M\downarrow)^!(X := p, \overline{X} := 1-p) = \frac{p}{1-q} = 1$$

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$\dots$	

$$[\![M]\!]^!(p, 1-p) = \mathbb{P}(M\downarrow)^!(X := p, \overline{X} := 1-p) = \frac{p}{1-q} = 1$$

$$\begin{aligned} \mathbf{t}^![\![M]\!](-\ln p, -\ln(1-p)) &= \mathbf{t}\mathbb{P}(M\downarrow)^!(X := -\ln p, \overline{X} := -\ln(1-p)) \\ &= \inf_{n\in\mathbb{N}}\{-\ln p - n\ln(1-p)\} = -\ln p \\ &= -\ln\Big(\sup_{\omega : M \twoheadrightarrow \star} \omega\Big) \end{aligned}$$

Tropical semantics makes the (infinite) search space finite!!

$M \twoheadrightarrow_X \star$	p
$M \twoheadrightarrow_{\overline{X}} M \twoheadrightarrow_X \star$	pq
$M \twoheadrightarrow_{\overline{X}} M \twoheadrightarrow_{\overline{X}} M \twoheadrightarrow_X \star$	pq^2
$\dots$	

$$[\![M]\!]^!(p, 1-p) = \mathbb{P}(M\downarrow)^!(X := p, \overline{X} := 1-p) = \frac{p}{1-q} = 1$$

$$\begin{aligned} \mathbf{t}^![\![M]\!](-\ln p, -\ln(1-p)) &= \mathbf{t}\mathbb{P}(M\downarrow)^!(X := -\ln p, \overline{X} := -\ln(1-p)) \\ &= \inf_{n\in\mathbb{N}}\{-\ln p - n\ln(1-p)\} = -\ln p \\ &= -\ln\left(\sup_{\omega : M \twoheadrightarrow \star} \omega\right) \end{aligned}$$

Theorem

For all terms $M : \mathrm{Bool}^n \to \mathrm{unit}$ *there exists an all-one formal* ***polynomial*** $p \in \mathbb{T}\{\mathbb{X}\}$ *such that*

$$\mathbf{t}^!\llbracket M \rrbracket = p^!$$

The ***tropical degree*** $\mathfrak{d}_M$ *of* M *is the minimum degree of such polynomials* p*.*

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For all terms $M : \mathrm{Bool}^n \to \mathrm{unit}$ *there exists an all-one formal* ***polynomial*** $p \in \mathbb{T}\{\mathbb{X}\}$ *such that*

$$\mathbf{t}^{!}[\![M]\!] = p^{!}$$

The ***tropical degree*** $\mathfrak{d}_M$ *of* M *is the minimum degree of such polynomials* p*.*

Theorem

Finding the tropical degree $\mathfrak{d}_M$ *for a term* $M : \mathrm{Bool}$ *is* Π^0_1*-hard.*

Theorem

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So

$$\mathfrak{d}_M = 1 \Leftrightarrow N \textit{ not normalisable.}$$

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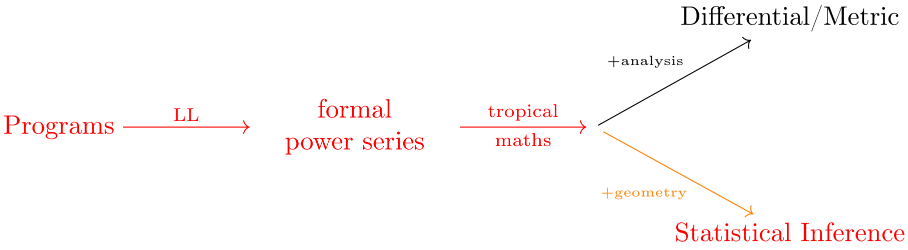
Ideally

Let $M : \mathrm{Bool}^n \to \mathrm{Bool}$ and $f : \mathbf{db} \to \mathcal{D}(\{0,1\})$.

$$f \text{ is } \epsilon\text{-DP} \implies [\![f; M]\!] \text{ is } \epsilon(\mathfrak{d}_M + 2)\text{-DP}$$


Programs
LL
formal
power series
tropical
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Differential/Metric
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Let $\vdash M : \mathrm{Bool}$. *Then*

$$M : \langle\, \Gamma \vdash^{p} n \,\rangle \quad \Longrightarrow \quad \mathbf{t}^{!}[\![M]\!] \leq p^{!}$$

and there is one with $\mathbf{t}^{!}[\![M]\!] = p^{!}$ *and* $\deg p = \mathfrak{d}_M$.

Problem: need to compute the tropical product $\Pi_i p_i$ **and minimise it**

Solution: Polytopes!

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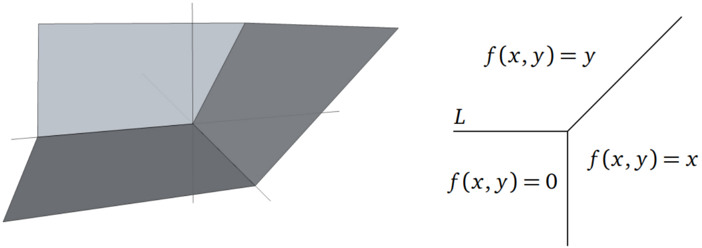

Figure 2.19: A tropical planar line

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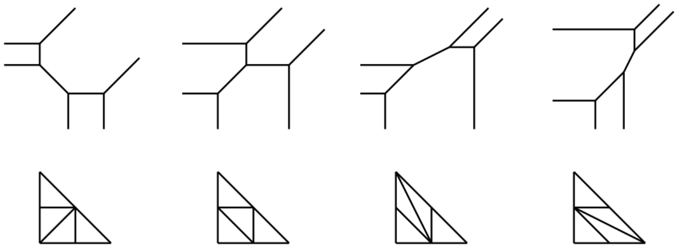

Figure 2.25: Smooth (non-degenerated) conics

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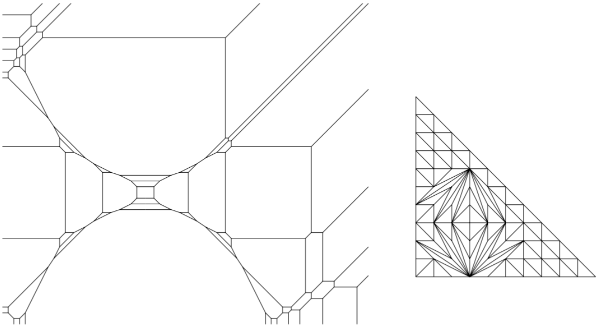

Figure 2.32: The Itenberg-Ragsdale curve of degree 10

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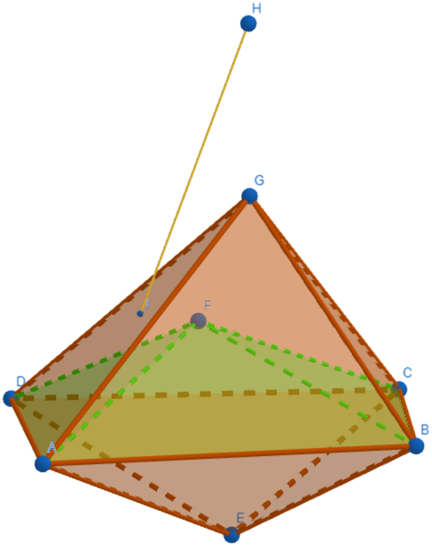

Problem: Newton(p) is for $p \in [-\infty, +\infty)\{X\}$. We work on $\mathbb{T} = [0, +\infty]$

Solution: Restrict Newton(p) to a sub-Polytope which is "minimal"

At a certain point we need an *obvious* property of the "visibility relation" of a facet from a point...

Merci & Grazie!