Interpolating Proof-Nets with Cuts

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Based jww. on Érin Le Boulc'h (IRIF), Guido Fiorillo (LIP)

Daniel Osorio Valencia (Torino) & Luc Pellissier (LACL)

Outline

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Statement of Craig's Interpolation Theorem

Definition (Interpolation property – Craig, 1957)

A logic L has the interpolation property if, for any formulas A, B such that $A \vdash_L B$, there is a formula C satisfying $\mathbb{V}\mathrm{oc}(C) \subseteq \mathbb{V}\mathrm{oc}(A) \cap \mathbb{V}\mathrm{oc}(B)$ and such that $A \vdash_L C$ and $C \vdash_L B$.

$\mathbb{V}\mathrm{oc}(C) \subseteq \mathbb{V}\mathrm{oc}(A) \cap \mathbb{V}\mathrm{oc}(B)$ means that:

(Craig) predicate symbols occurring in C occur both in A and B.

(Lyndon) predicate symbols occurring positively (resp. negatively) in C occur positively (resp. negatively) in both A and B. ($\mathbb{V}\mathrm{oc} \mapsto \mathbb{V}\mathrm{oc}^{+}$ / $\mathbb{V}\mathrm{oc}^{-}$.)

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Remark

- *Ex: $p \wedge (q \wedge r) \vdash q \vee (s \vee p)$ can be interpolated by both p and q.*
- *Many applications in mathematical logic and CS (Beth definability, model-checking, query planning in DB...)*
- *Logical entailment can be factored through the interpolant, which amounts to* **introducing a cut***:*

$$\frac{A \vdash_L C \qquad C \vdash_L B}{A \vdash_L B}\ (\mathrm{Cut})$$

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Maehara's method (1960):

- For a provable sequent $A \vdash B$, consider (i) a cut-free derivation of $A \vdash B$ in order to exploit (ii) the inductive structure the cut-free derivation and (iii) the subformula property.
- Strengthen the induction hypothesis, by showing that if $\Gamma \vdash \Delta$, then for any splitting Γ', Γ'' of Γ and Δ', Δ'' of Δ there exists an interpolant C with $\mathbb{V}\mathrm{oc}(C) \subseteq \mathbb{V}\mathrm{oc}(\Gamma', \Delta') \cap \mathbb{V}\mathrm{oc}(\Gamma'', \Delta'')$ such that $\Gamma' \vdash C, \Delta'$ and $\Gamma'', C \vdash \Delta''$.

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- Maehara' method is of wide applicability for sequent calculi with cut-admissibility:
  - Maehara (1960) for LK, extended to LJ by Schütte (1962);
  - Prawitz adapted it to natural deduction (1965);
  - Roorda analysed for which fragments of LL interpolation holds (1994) *(does not cover the exponentials actually...)*;
  - Various modal logics;

Proof-Relevant Interpolation as Cut-Introduction

Refining Maehara's method

Prove that for any splitting of the conclusion sequent, one can find (i) an interpolant formula and (ii) two interpolating proofs such that cutting together the interpolating proofs cut-reduces to the original proof.

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Interpolation as Cut-Introduction proceeds in two phases:

- Bottom-up phase: Given a splitting of the root sequent, propagate it through π from root to axioms, splitting each visited sequent Γ, as (Γ', Γ''), according to the ancestor relation. In the end, there are 4 cases for an axiom rule $\vdash A^\perp, A$ (similarly for $\top$ axioms):

$$\vdash A^\perp, A \qquad \vdash A^\perp, A \qquad \vdash A^\perp, A \qquad \vdash A^\perp, A$$

- Top-down phase: From split-sequents, apply cut-expansion rules starting from axioms, moving the cuts down in such a way, ultimately, as to reach the root sequent of the original proof.

Proof-Relevant Interpolation Situations...

Proof-Relevant Interpolation as Cut-Introduction

Theorem

Let Γ, Δ *be lists of* LL *formulas and* $\pi \vdash \Gamma, \Delta$ *be cut-free.*
There exists a LL *formula* C *such that* $\mathbb{V}\mathrm{oc}^{+}(C) \subseteq \mathbb{V}\mathrm{oc}^{-}(\Gamma) \cap \mathbb{V}\mathrm{oc}^{+}(\Delta)$ *and* $\mathbb{V}\mathrm{oc}^{-}(C) \subseteq \mathbb{V}\mathrm{oc}^{+}(\Gamma) \cap \mathbb{V}\mathrm{oc}^{-}(\Delta)$ *and two cut-free proofs* π_1, π_2 *of* $\vdash \Gamma, C$ *and* $\vdash C^{\perp}, \Delta$ *respectively such that*

$$\dfrac{\overset{\pi_1}{\vdash \Gamma, C} \qquad \overset{\pi_2}{\vdash C^{\perp}, \Delta}}{\vdash \Gamma, \Delta}\ (\mathrm{Cut}) \quad \longrightarrow^{\star}_{\mathrm{cut}} \ \pi.$$

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Theorem established using PRIS:

$$\dfrac{\overset{\pi_1^L}{\vdash \Gamma_1, I_1} \qquad \overset{\pi_1^R}{\vdash I_1^{\perp}, \Delta_1}}{\vdash \Gamma_1, \Delta_1}\ (\mathrm{Cut}) \quad \dots \quad \dfrac{\overset{\pi_n^L}{\vdash \Gamma_n, I_n} \qquad \overset{\pi_n^R}{\vdash I_n^{\perp}, \Delta_n}}{\vdash \Gamma_n, \Delta_n}\ (\mathrm{Cut})$$

$$\pi$$

$$\vdash \Gamma, \Delta$$

Related proof-relevant Interpolation results

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Prawitz:

LEMMA. *Let* Π *be a normal deduction in* C' *of* A *depending on* Γ, *and let* Γ_1 *and* Γ_2 *be two disjoint sets such that* $\Gamma_1 \cup \Gamma_2 = \Gamma$. *Then there is a formula* F, *called an interpolation formula to* $\langle \Gamma_1, \langle \Gamma_2, A \rangle \rangle$, *such that* $\Gamma_1 \vdash F$ *and* $\{F\} \cup \Gamma_2 \vdash A$ *and such that every parameter that occurs positively [negatively] in* F *occurs positively [negatively] in some formula of* Γ_1 *and negatively [positively] in some formula of* $\Gamma_2 \cup \{\sim A\}$.

(for NK and NJ)

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(for NK and NJ)

Čubrić: Similar proof-relevant result established in 1993 for simply typed λ-calculus by refining Prawitz's result, in a paper that was almost never cited:

Proposition 3.3. *Let L_1 and L_2 be two languages, and let T_1 and T_2 be two $\lambda\delta$-theories on the respective languages. Let T_0 be a theory on the language $L_1 \cap L_2$ such that $T_0 \subset T_1 \cap T_2$ (we may as well assume that the theories are deductively closed). Let $(x^B \triangleright t^C)$ be a term in the language $L_1 \cup L_2$ such that the type B is in L_1 and the type C is in L_2. Then, there is a type A in $L_1 \cap L_2$ and terms $(x^B \triangleright r^A)$ in L_1, and $(y^A \triangleright s^C)$ in L_2 such that:*

$$T_1 \cup T_2 \vdash t =_{x^B} s(r/y).$$

Related proof-relevant Interpolation results

Other related works:

- Matthes (2001) extended Čubrić results to a λ-calculus with generalized elimination rules.
- Recent related work for substructural logics, e.g. Lambek calculus by Veltri and Wan.

PR-interpolation can also be extended:

- In natural deduction style:
  - to some modal logics, e.g. S4;
  - to Parigot's $\lambda\mu$-calculus (with Guido Fiorillo);
  - using bidirectional typing to revisit Čubrić' result (with Meven Lennon-Bertrand);
  - trivial for System F thanks to the ability to abstract over types (leading to a uniform computational interpolation).
- In sequent style:
  - to circular proofs for μ-calculi;
  - to System L.

Pushing the limits of the proof-relevant approach

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Pushing the limits of the proof-relevant approach

Maehara's approach strongly relies of the inductive structure of sequent proofs:

- trees: they have a tree structure. Proof Nets
- assumed to be cut-free; Proofs with (some) cuts

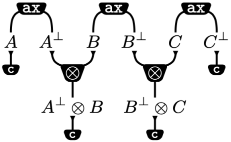
What happens when those two assumptions are weakened?

Outline

Proof-net PR Interpolation via Parsing Nets

Proof-nets are graph-like proof objects obtained by forgetting the order of application of inferences from sequent proofs.

Canonical, good properties (e.g. confluent cut-elimination) but the logical correctness is not local anymore:



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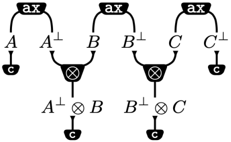

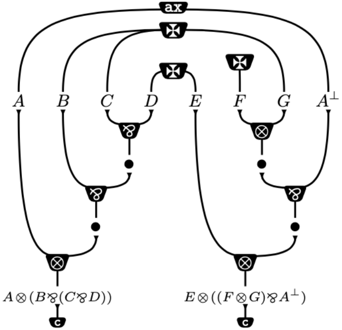

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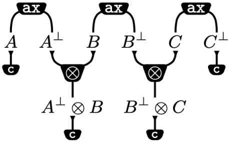
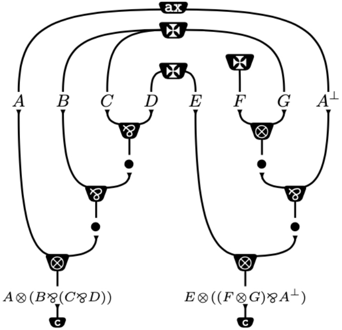
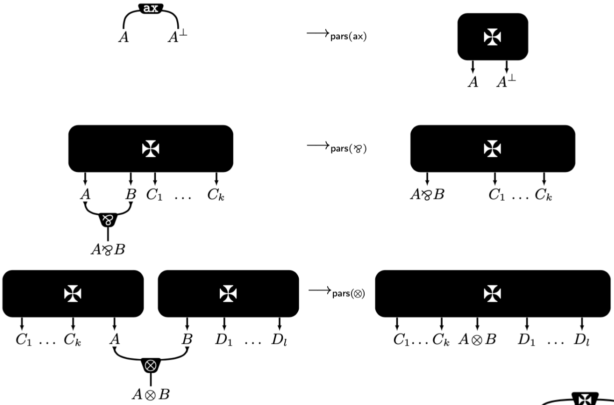
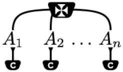




How do we check whether a given graph corresponds to a (correct) sequent proofs: **correctness criteria**

Proof-net PR Interpolation via Parsing Nets





A proof structure is correct iff it contracts to:

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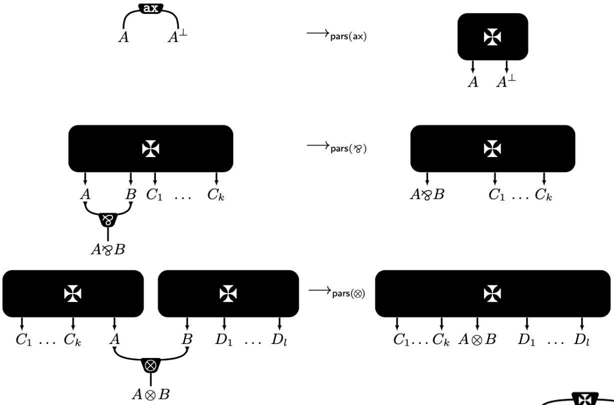

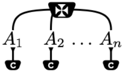

A proof structure is correct iff it contracts to:

(This can be extended to a more general graph rewriting which does not proceed top-down: contractiblity.)

Extracting logical information from correctness criteria

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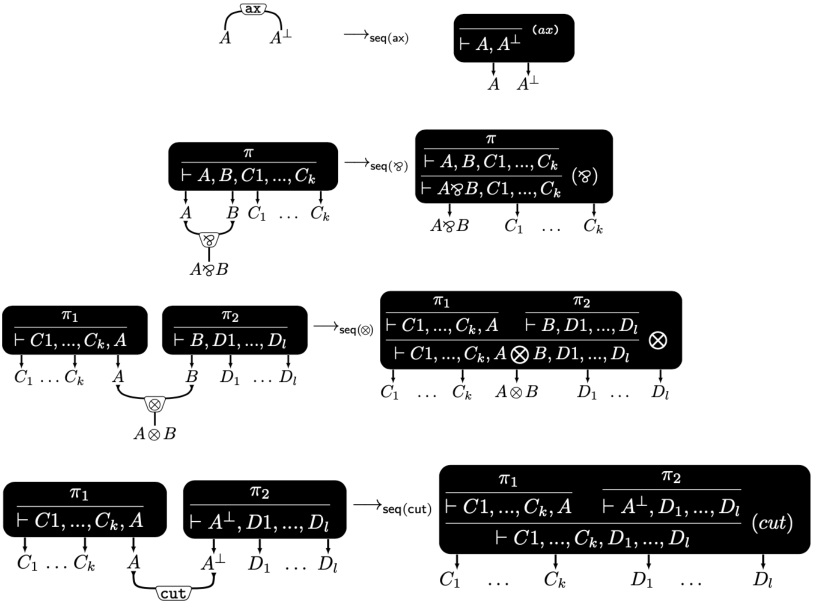
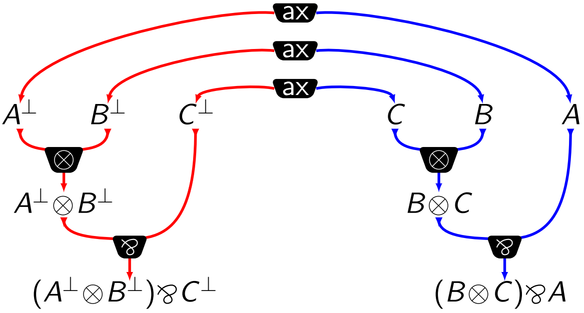


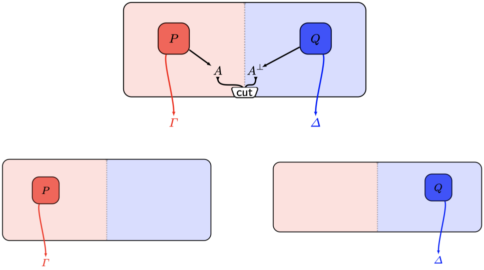
Interpolation by Parsing: colouring

First step : colour bottom-up the proof net



Interpolation by Parsing: Interpolation box

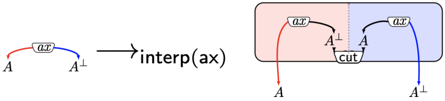
One can replace the ✠ with a more informative labelling, an interpolation box:

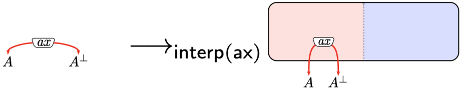


Using units

One can interpolate unit-free MLL in MLL with units and jumps avoiding the need to use half-empty interpolation boxes.

Interpolation by Parsing





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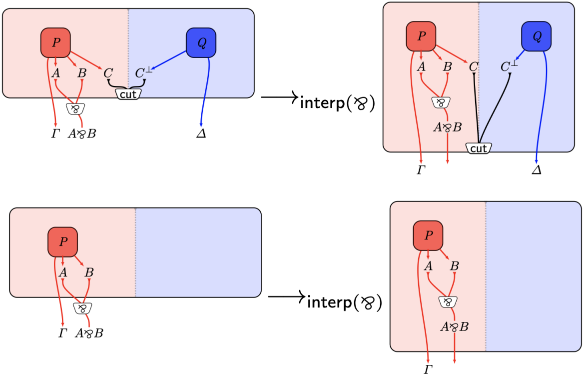

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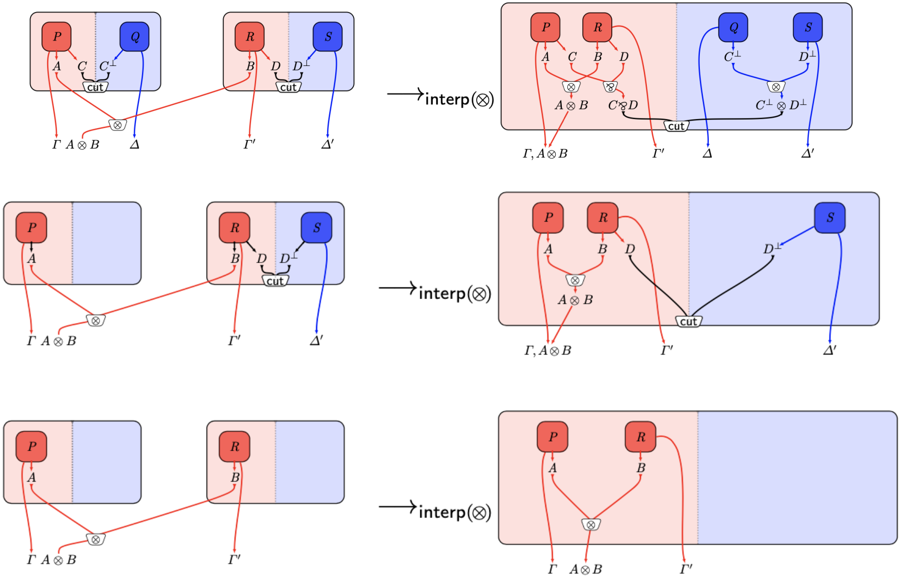
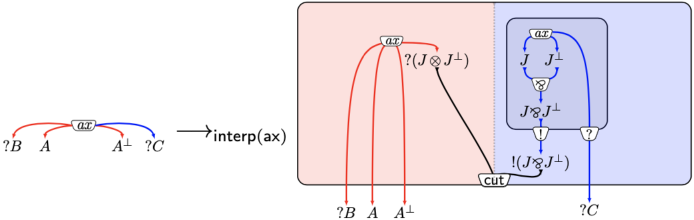


Interpolating MELL proof nets



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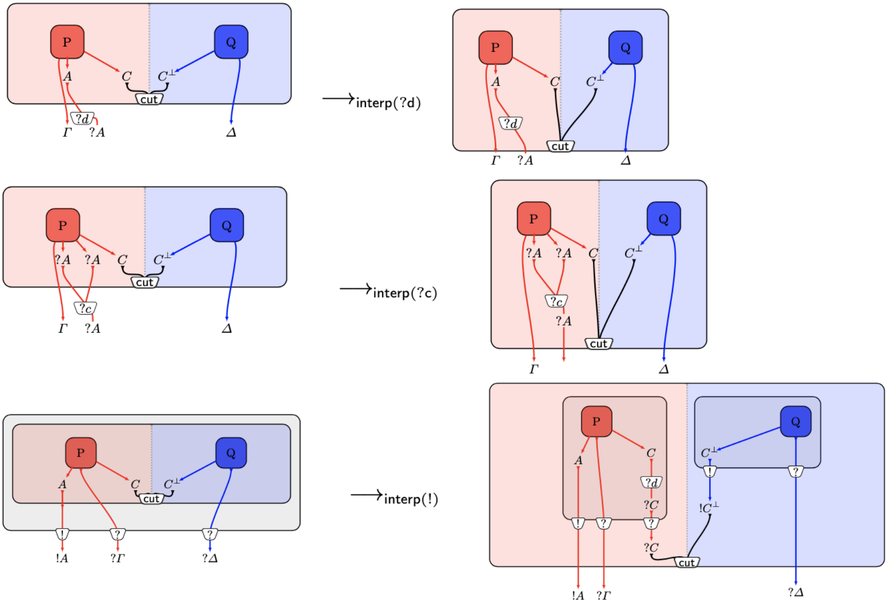

Different interpolants

- Maehara, Prawitz Methods $\rightarrow$ One interpolant
- Interpolation by parsing $\rightarrow$ Different interpolants based on the choice of rewriting steps.
- What is the relation between interpolants coming from different sequentialisations?

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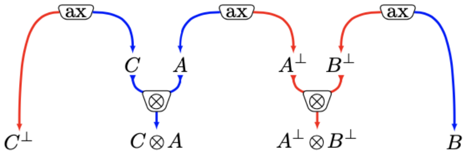
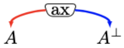


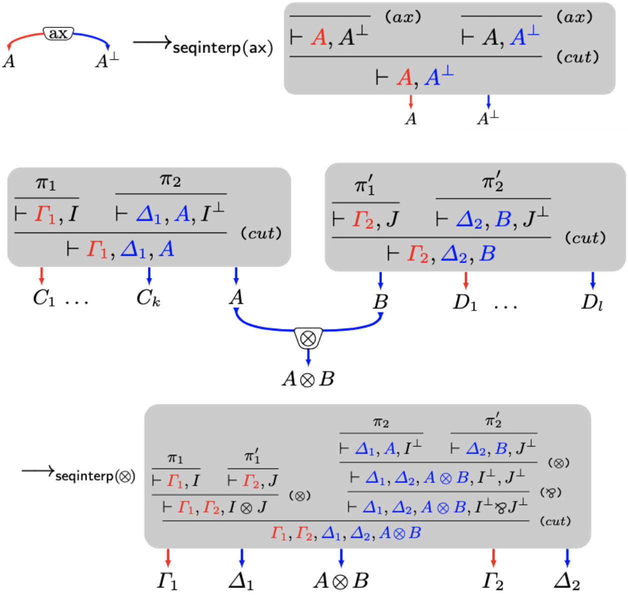
Combining sequentialisation and interpolation



$$\longrightarrow_{\mathsf{seqinterp(ax)}} \quad \dfrac{\dfrac{}{\vdash A, A^{\perp}}\,(ax) \qquad \dfrac{}{\vdash A, A^{\perp}}\,(ax)}{\vdash A, A^{\perp}}\,(cut)$$

$A \qquad A^{\perp}$

Combining sequentialisation and interpolation



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What is the difference wrt. first sequentialising and then performing PR sequent interpolation?

- Going through PN and parsing led us to avoid useless introduction of units;
- With our approach, the parsing sequence can be built by exploiting the current partial interpolants (for instance to optimise size of interpolant, logical strength, minimise polarity shifts, etc.);
- There is more than just parsing in exploiting correctness criteria for interpolation.

Outline

Proof-relevant Interpolation with cuts

Extending the method with cuts?

- The cut rule is the only inference breaking the subformula property: Maehara's method relies on cut-freeness;
- Though, avoiding full cut-elimination would be beneficial (usually induces size blow up, possibly partial cut-elimination result only, etc.);
- Use the *(standard)* trick of viewing a cut on C as an introduction of $C \otimes C^{\perp}$ followed by a hiding operation
- The decoration of the proofs based on a splitting is straightforwardly extended to proofs with cuts: cuts are given an arbitrary color (ie the two cut formulas have the same color).

Introduce the notion of cut-vocabularies.

Proof-relevant Interpolation with cuts

...under suitable assumptions

Theorem (Maehara-Čubrić lemma with cuts)

Let S be a PN of conclusions Γ, Δ. Let $\mathbb{C} = \{(C_1, C_1^\perp), \ldots, (C_n, C_n^\perp)\}$ be the cut formulas of S, let L, R be any partitioning of $\{1, \ldots, n\}$ and $\mathbb{C}^L = \mathbb{V}\mathrm{oc}(C_i, i \in L)$ and $\mathbb{C}^R = \mathbb{V}\mathrm{oc}(C_i, i \in R)$. Then there exist

- *a formula I s.t. $\mathbb{V}\mathrm{oc}(I) \subseteq (\mathbb{V}\mathrm{oc}(\Gamma) \cup \mathbb{C}^L) \cap (\mathbb{V}\mathrm{oc}(\Delta) \cup \mathbb{C}^R)$ and*
- *two proof nets S_1, S_2 of conclusions Γ, I and $I^\perp, \Delta$ respectively*

s.t. the proof net obtained by cutting S_1 and S_2 on $I, I^\perp$ reduces to S.

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- As such, not an interpolation result...
- but allows to analyse conditions on the cut-vocabulary to obtain interpolation!

Proof-relevant Interpolation with cuts

...under suitable assumptions

Theorem (Interpolation theorem with cuts)

Let S be a PN of conclusions Γ, Δ. Let $\mathbb{C} = \{(C_1, C_1^\perp), \ldots, (C_n, C_n^\perp)\}$ be the cut formulas of S, let L, R be any partitioning of $\{1, \ldots, n\}$ and $\mathbb{C}^L = \mathbb{V}\mathrm{oc}(C_i, i \in L)$ and $\mathbb{C}^R = \mathbb{V}\mathrm{oc}(C_i, i \in R)$ such that

- $(\mathbb{C}^L \setminus \mathbb{V}\mathrm{oc}(\Gamma)) \cap (\mathbb{V}\mathrm{oc}(\Delta) \cup \mathbb{C}^R) = \varnothing$ *and*
- $(\mathbb{C}^R \setminus \mathbb{V}\mathrm{oc}(\Delta)) \cap (\mathbb{V}\mathrm{oc}(\Gamma) \cup \mathbb{C}^L) = \varnothing$.

Then there exist

- *a formula I such that $\mathbb{V}\mathrm{oc}(I) \subseteq \mathbb{V}\mathrm{oc}(\Gamma) \cap \mathbb{V}\mathrm{oc}(\Delta)$ and*
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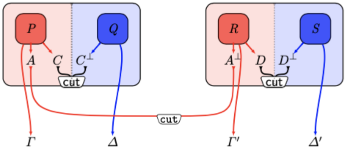
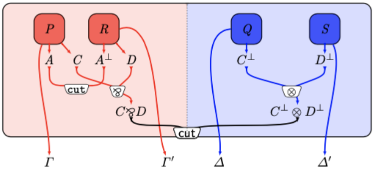
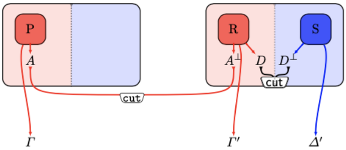
Proof-net Interpolation by Parsing, with Cuts

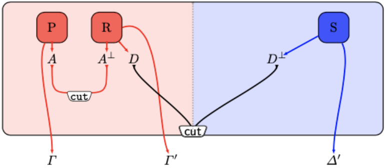


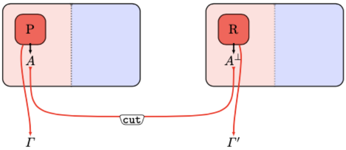
$\longrightarrow$interp(cut)



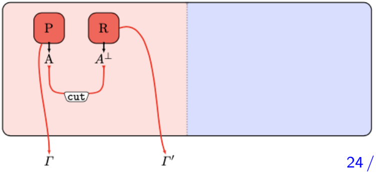


$\longrightarrow$interp(cut)





$\longrightarrow$interp(cut)



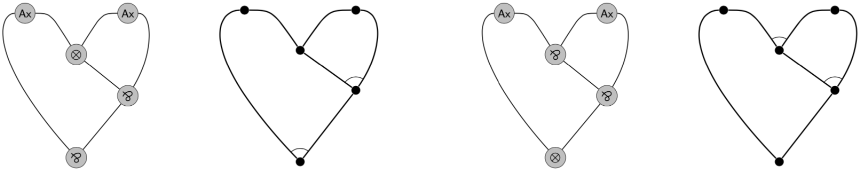
Outline

Interpolating Proof-Nets with Contractibility

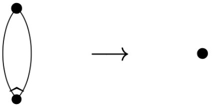
- Parsing proceeds top-down, from axioms to conclusions: it relies on a similar inductive structure as sequentialised proofs
- Contractibility is generalisation of parsing where contractions can occur anywhere in the proof net
- Interpolation using contractibility graphs labelled with (pairs of) interpolation modules

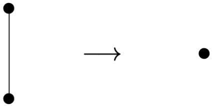
Interpolating Proof-Nets with Contractibility

- Paired graph:



- Contraction rules:

R_1 : 

R_2 : 

(A contraction rule can be applied to distinct nodes only.)

Theorem

A para-proof structure is a para-proof net if, and only if, its paired graph can be contracted to a single point applying rules R_1 and R_2.

Interpolating Proof-Nets with Contractibility

- We consider contractibility graphs having nodes labeled with pairs of "interpolation modules";
- The contraction rules will graft the interpolation modules together;
- In the current state, we use units to treat the mono-color axiom case.

Interpolating Proof-Nets with Contractibility

- We consider contractibility graphs having nodes labeled with pairs of "interpolation modules";
- The contraction rules will graft the interpolation modules together;
- In the current state, we use units to treat the mono-color axiom case.
- BLACKBOARD

Outline

Conclusion

Summary

- Cut-introduction synthesises proof-relevant interpolants.
- Interpolant as an interface datatype through which a given computation can be factored.
- Čubrić' work overlooked: Hope this can bring some interest to his work!
- Usual assumptions can be weakened: the technique still works with cuts of some form and can be lifted to proof nets.

Ongoing and future works

- Computational interpretation of uniform interpolation? (Trivial for F, no clue in other settings.)
- Extend the PR interpolation for proof nets to other criteria.
- Semantical interpolation result (ongoing with Fiorillo) $+$ Game-semantical interpretations of the result.
- Characterizing more precisely the kind of cuts which can be considered in the interpolation process could be useful: System L, Hetzl and Jalali results, etc.
- Extend Čubrić' interpolation to non normal λ-terms: via proof-nets or bidirectional typing.
- Impact of the PR approach on other applications of interpolation (e.g. in database theory or model-checking)?