Oldies but goodies?

Proof nets as edge-coloured graphs

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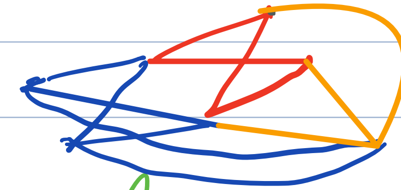
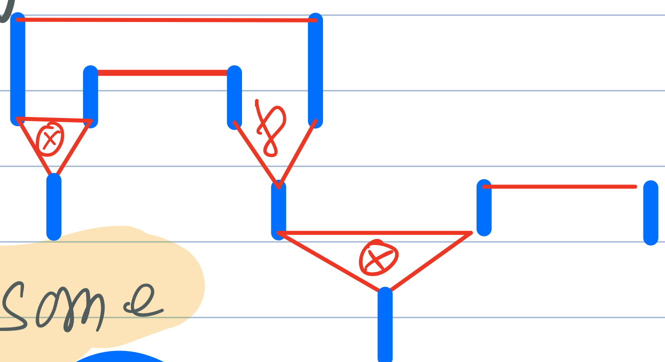
Handwritten slides:
- no sources, too old...
- AIs ain't as good (yet?) as I've been told ;-)

Aggregates 1992



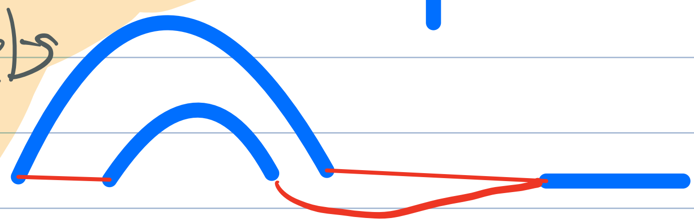
complete bipartite graphs, one colour for each

Edge bicoloured w links 1994



no ea cycle

Handsome proofnets 1996



every ea cycle contains a chord

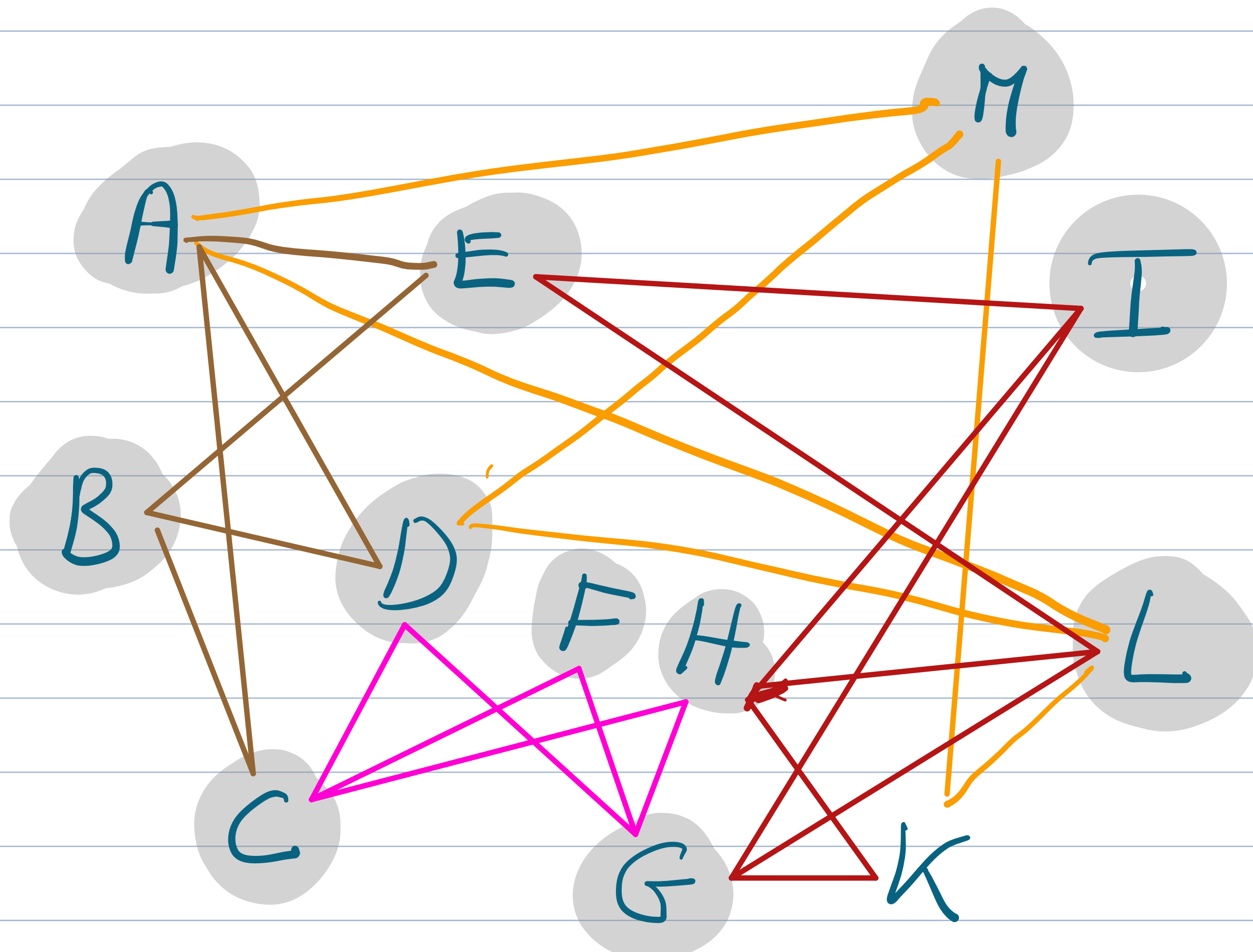
Cograph (formula) + perfect matching (axioms)

same techiques → towards a sequent calculus for POMSET LOGIC

AGGREGATE

- edge coloured graph
- colour = complete bipartite graph

$K_{n,p}$



Heterochromatic $\begin{cases} \text{path} \\ \text{set of edges} \end{cases}$ Γ

(bigarré)

variegated

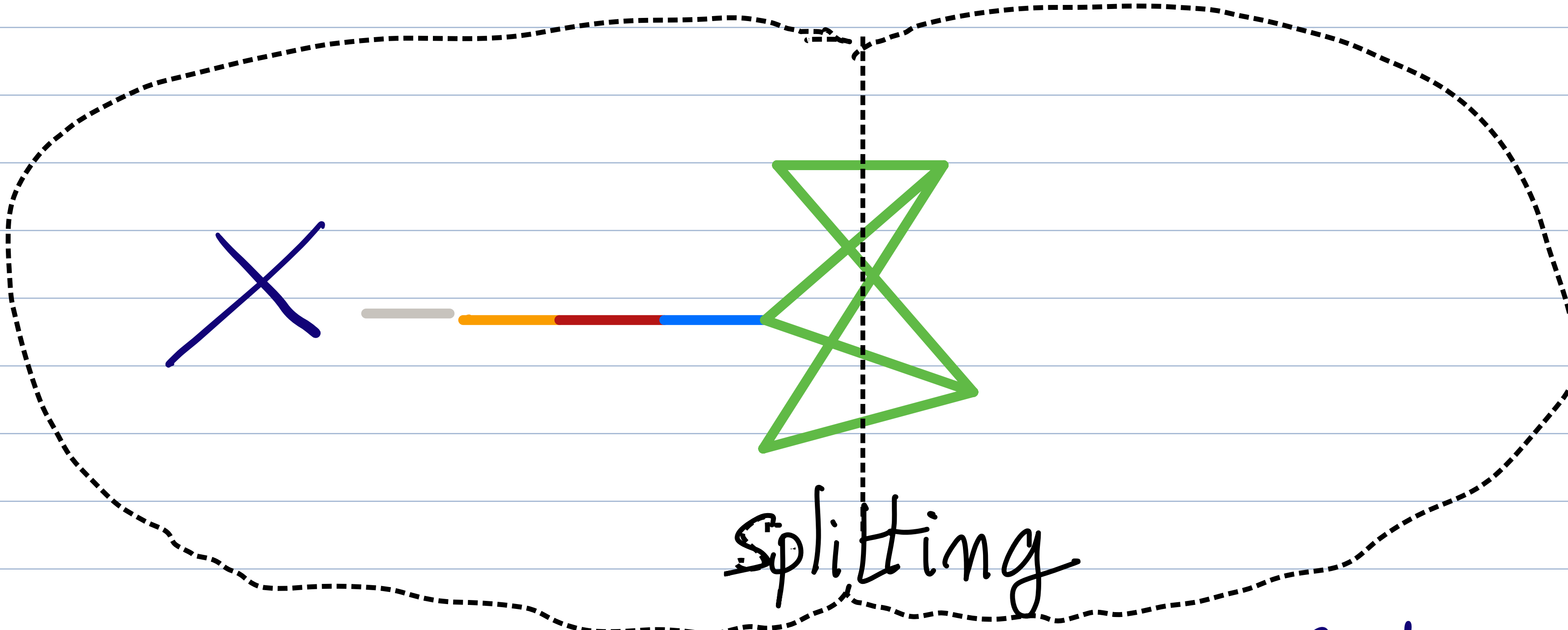
For each colour α

at most one α-edge in Γ

Correct: no heterochromatic cycle

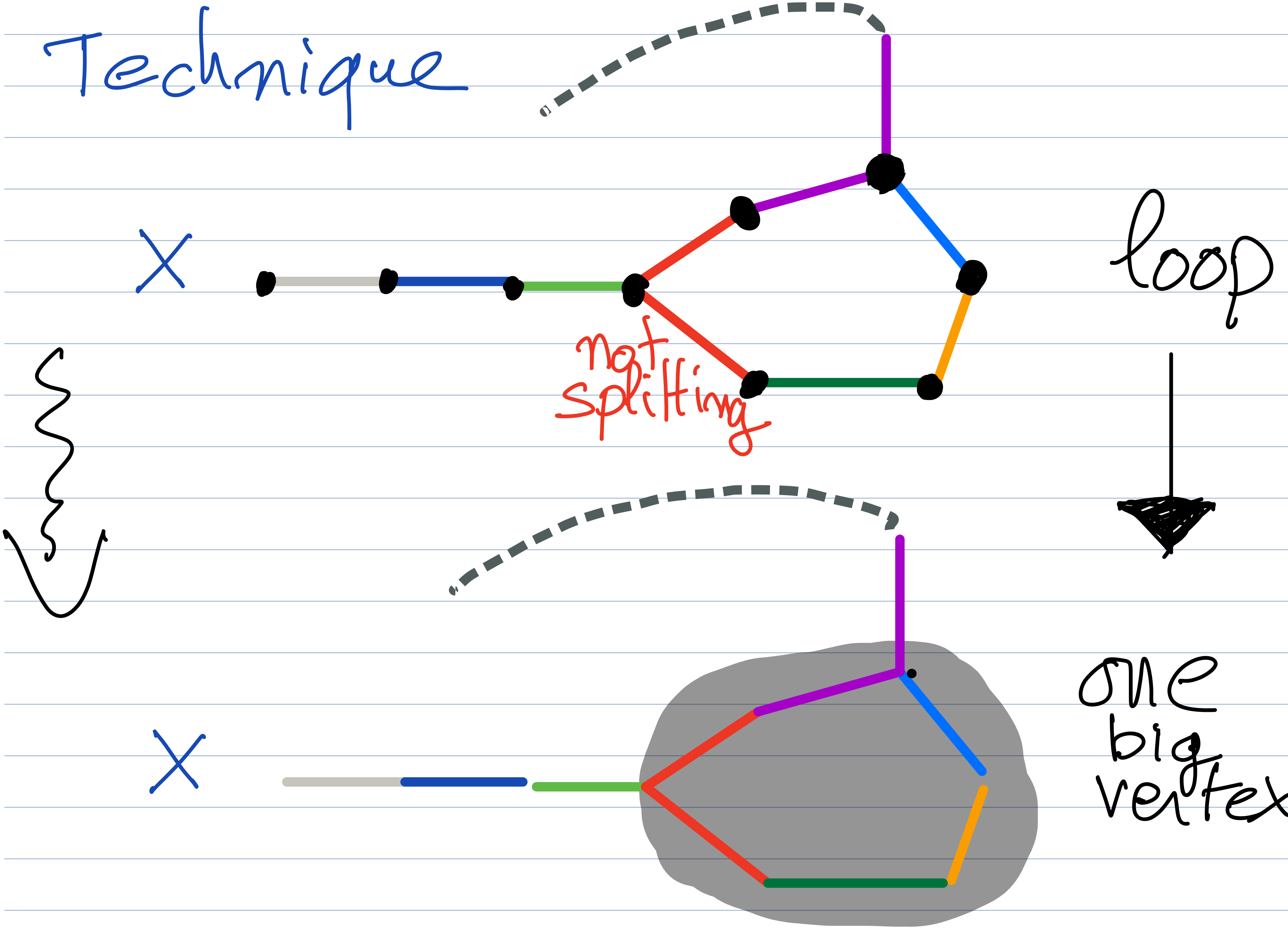
No Mix rule: an heterochromatic path between any two vertices

Result for all X — H-path to a H-cycle



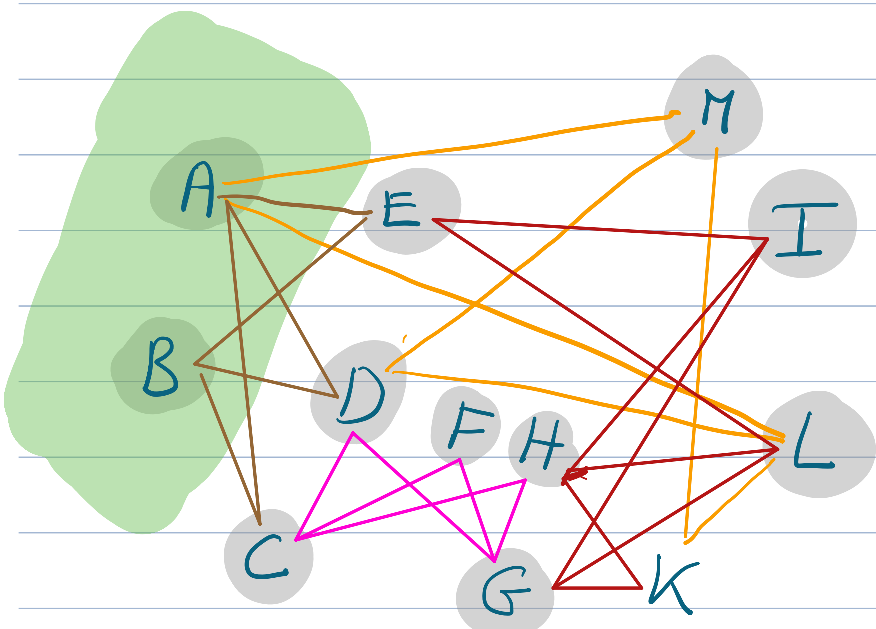
or H path to a splitting K_{4p}

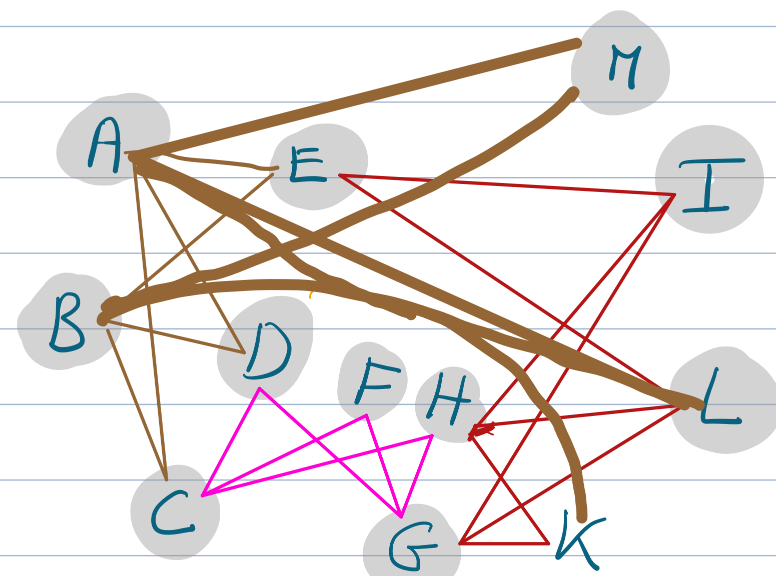
Technique



H paths are preserved can be reconstructed stretching

graph transformation preserving H path in both directions





preserves H paths, splitting K_{mp}'s

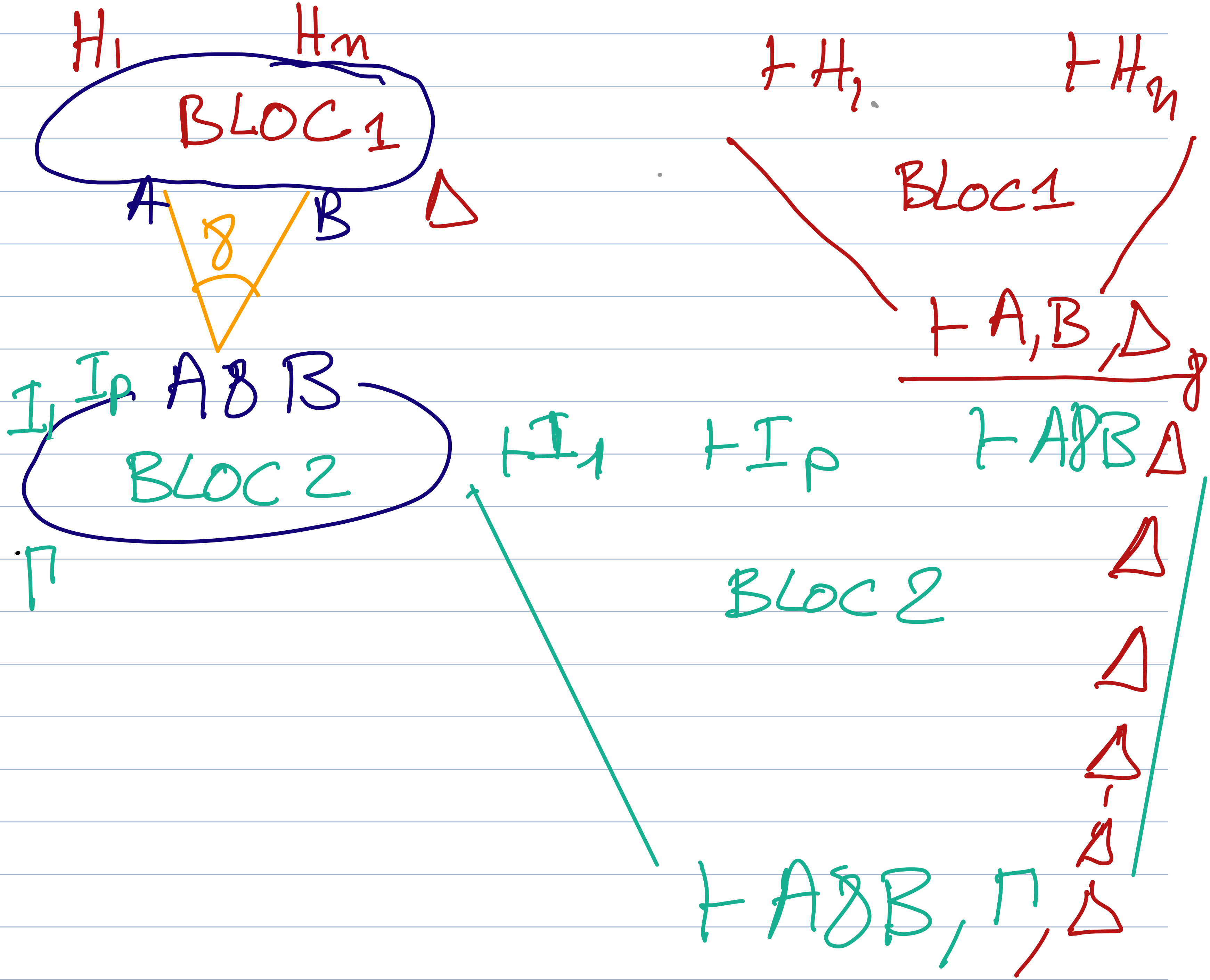
Sequentialisation à la Danos

Splitting ⅋

blocs: CC with { axioms
times links

par links ($K_{1,2}$ or $K_{1,1}$)

in between blocs.



H1
Hn
BLOC 1
A
B
Δ
γ
I1
Ip
AγB
BLOC 2
Γ
⊢ H1
⊢ Hn
BLoc1
⊢ A, B, Δ
γ
⊢ I1
⊢ Ip
⊢ AγB
Δ
BLoc 2
Δ
Δ
Δ
⊢ AγB, Γ, Δ

If A and B in two different blocs

$$\vdash \Pi_1 \quad \vdash \Pi_i \qquad\qquad \vdash \Pi_{i+1} \cdots \vdash \Pi_n$$

$$\text{BLOC 1.a} \qquad\qquad \text{BLOC 1.b}$$

$$\dfrac{\vdash A, \Delta_a \qquad\qquad \vdash B, \Delta_b}{\vdash A \otimes B, \Delta_a, \Delta_b}\ \text{Mix}$$

When any two vertices are connected by an Π path
→ it never happens.

Splitting times (à la Girard)

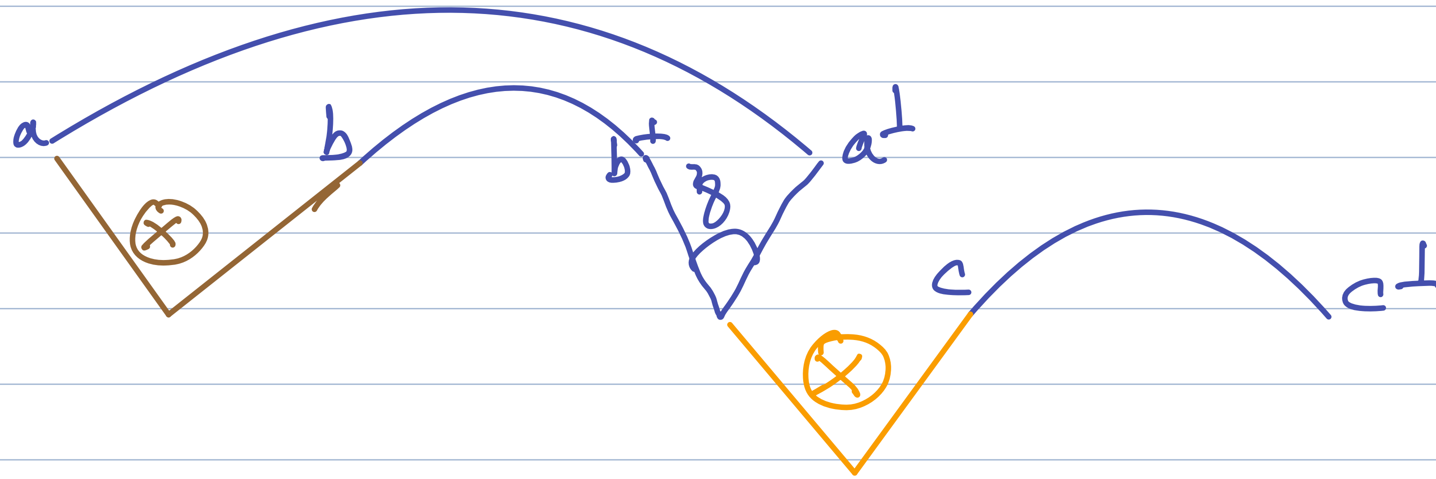
vertices : axioms (weird.)

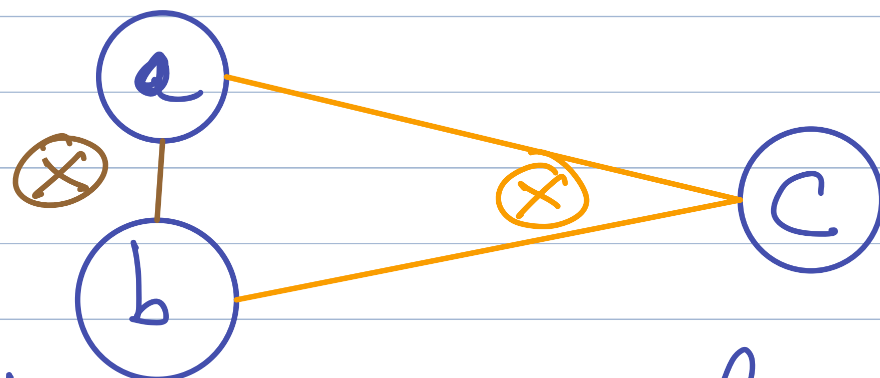
colours : $(A \otimes_\alpha B)$

an α-edge from $a\ a^\perp$ to $b\ b^\perp$

when $(a \cap a^\perp)$ and $(b \cap b^\perp)$

are one above (in) A
one above (in) B

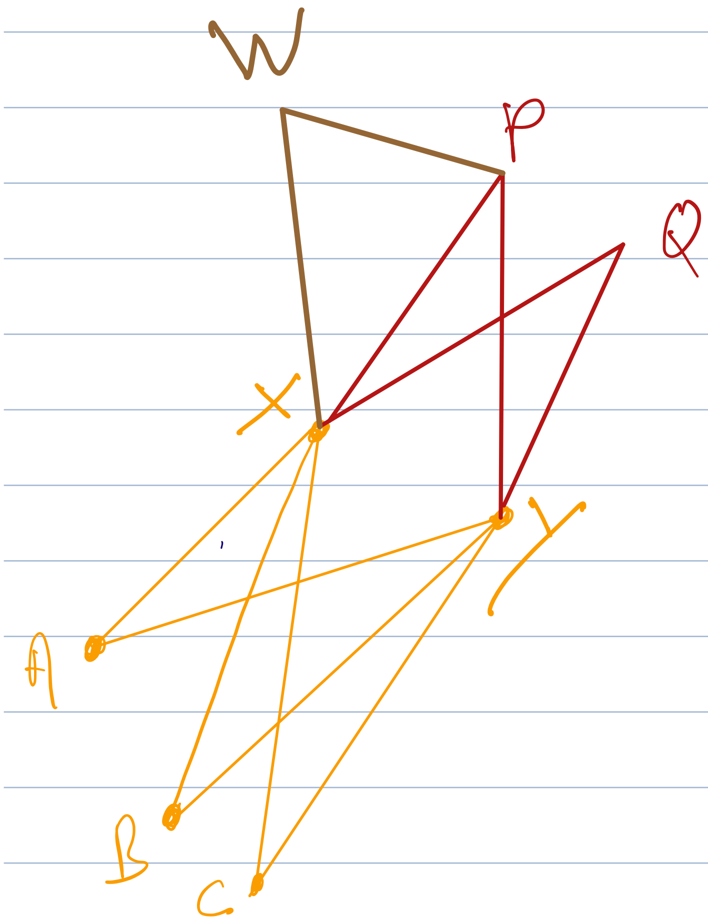


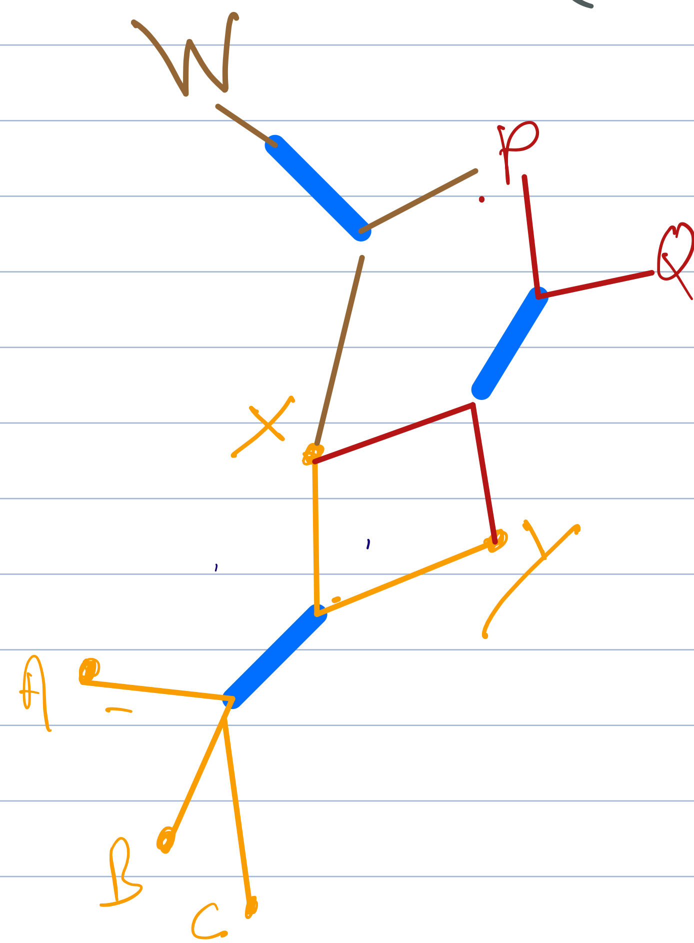


H paths ~ Danos feasible paths

no H cycle $\longrightarrow$ splitting times playing with graphs in Sophia

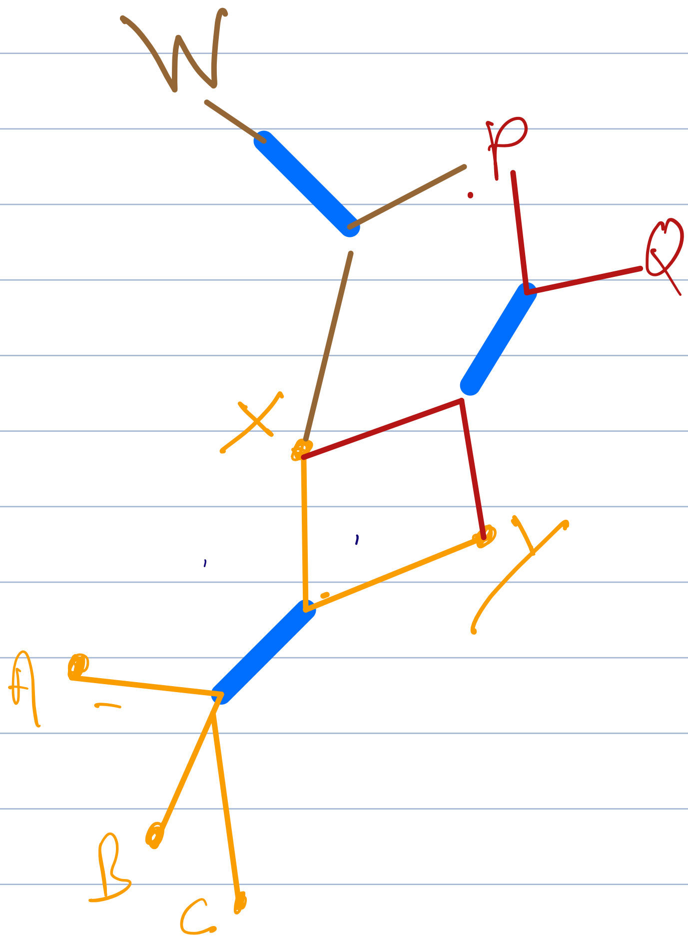
(1994)

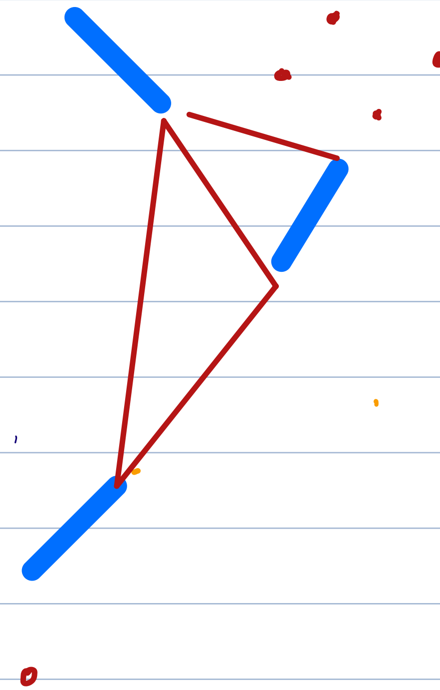




thanks to Jean Claude BERMOND

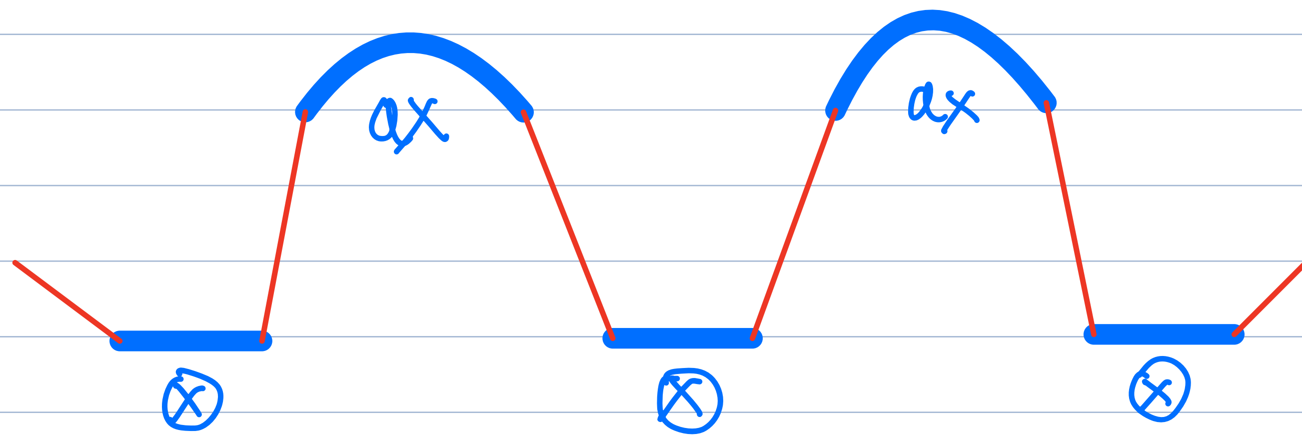
Two colours





H paths $\longrightarrow$ alternate elementary path.

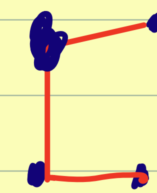
axioms ⊗ links.

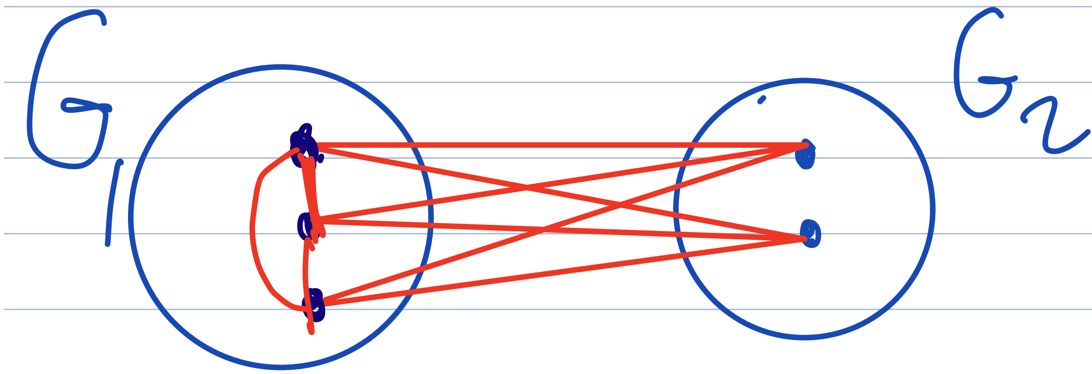


2 colours are enough

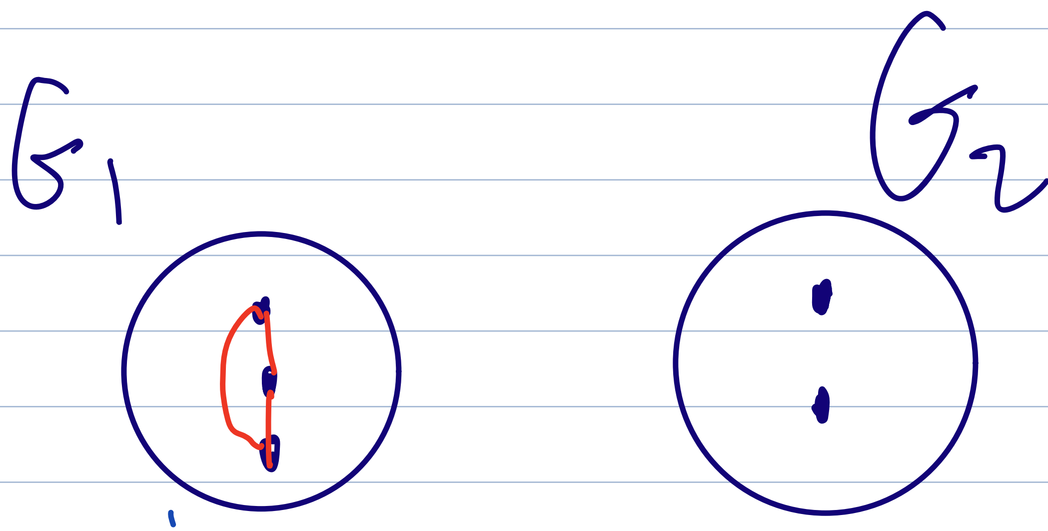
suppress axioms $\longrightarrow$ back to aggregates

suppress ⊗ $\longrightarrow$ handsome proof nets cographs (series parallel graphs)

P_4 - free 



$G_1 \otimes G_2$

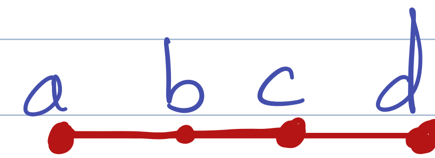


$G_1 \parr G_2$

$\oplus$

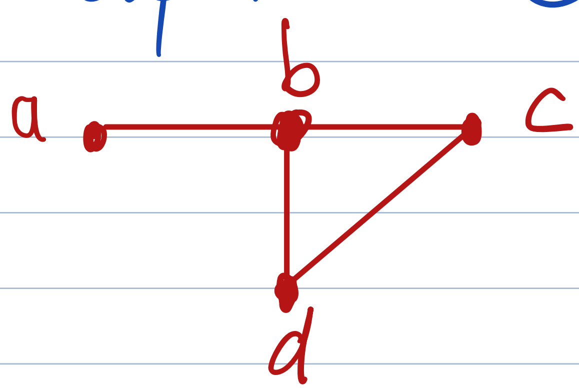
Cographs

- characterisation

P_4 - free $\quad G|_{a,b,c,d} \neq$ 

;

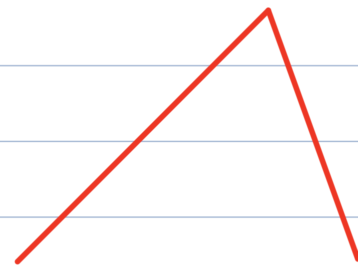
- decribed by a unique term up to $\otimes$ $\parr$ associative, commutative



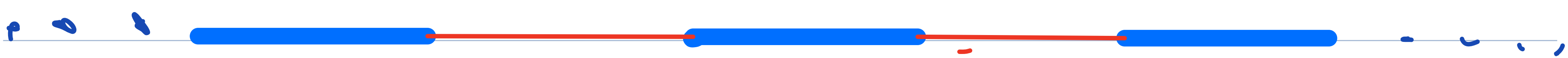
$b \otimes (a \parr (d \otimes c))$

Graph B edges perfect matching

R edges as you like



alternate elementary path

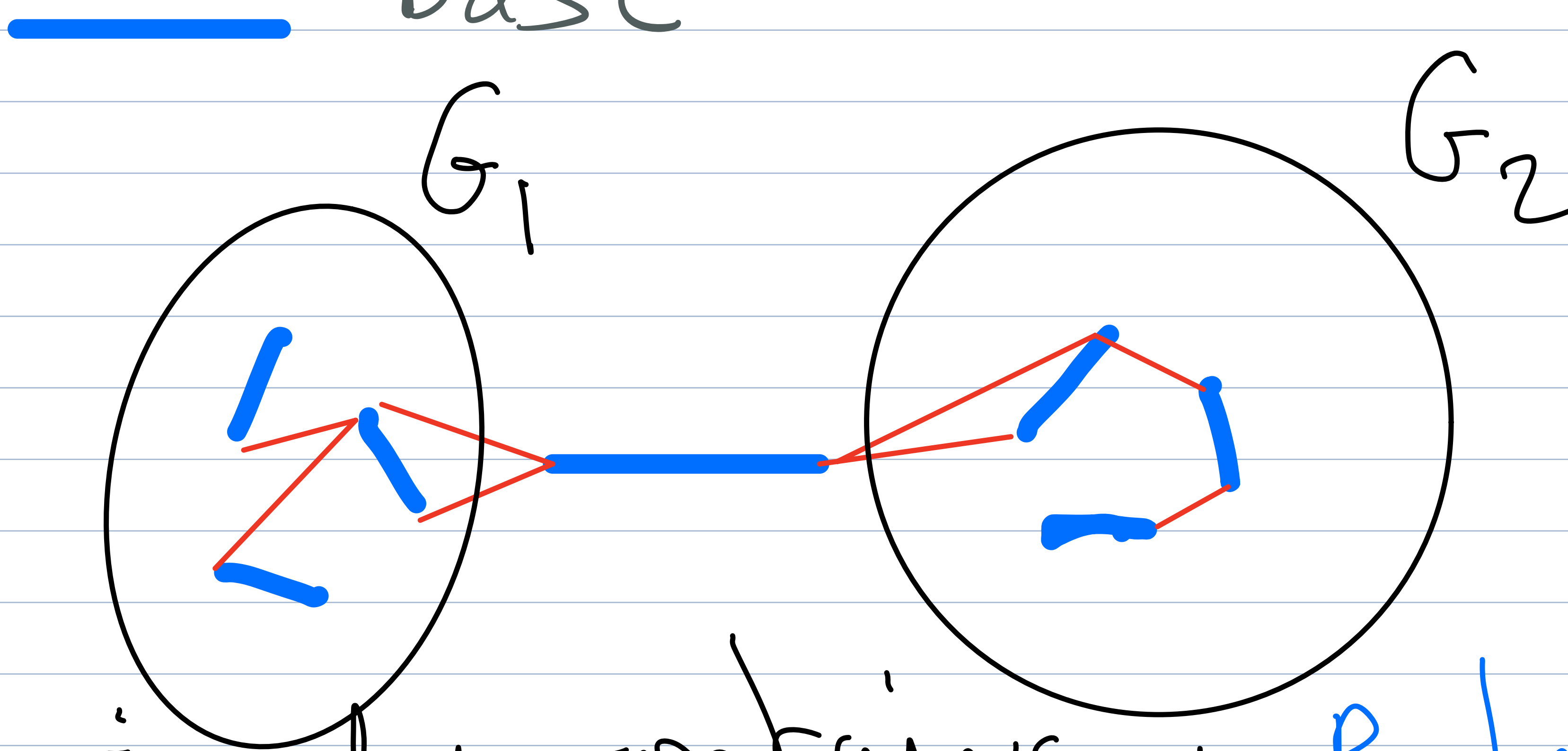


ae paths / ae cycle

ae cycle-free [unique perfect matching

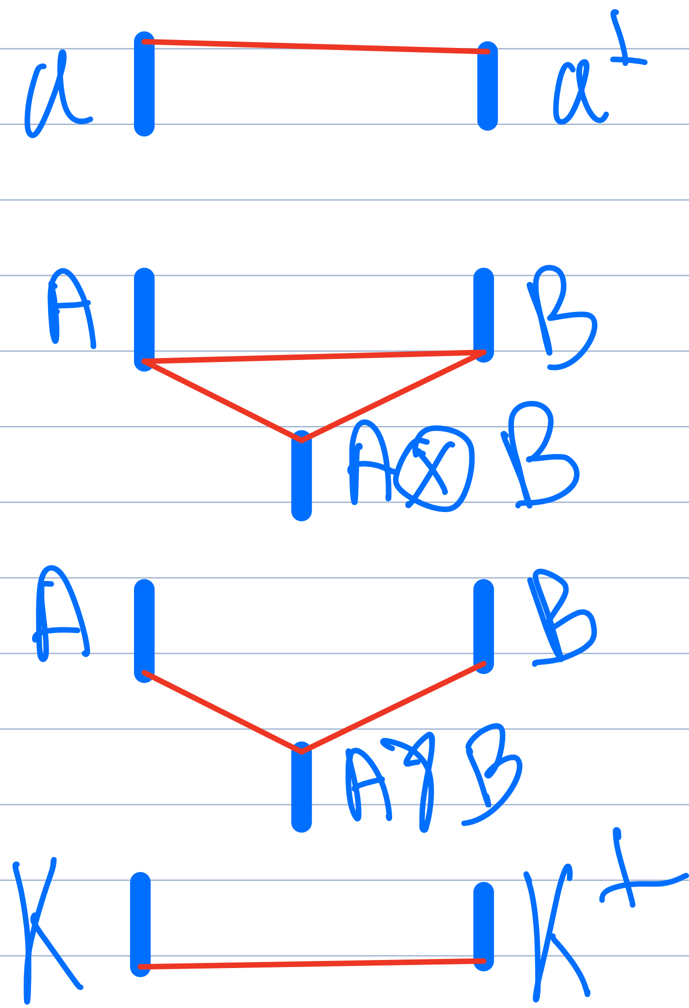
No ae-cycle iff in this inductive class of graphs:

base



recursively contains a B-bridge

Links Danos Regnier style formulas: B-edges



axiom

⊗

⅋

cut

criterion:

$$\frac{\text{no ae-cycle}}{\text{stable by cut elim.}}$$

DR cycle-free.

$\longleftrightarrow$ no ae-cycle

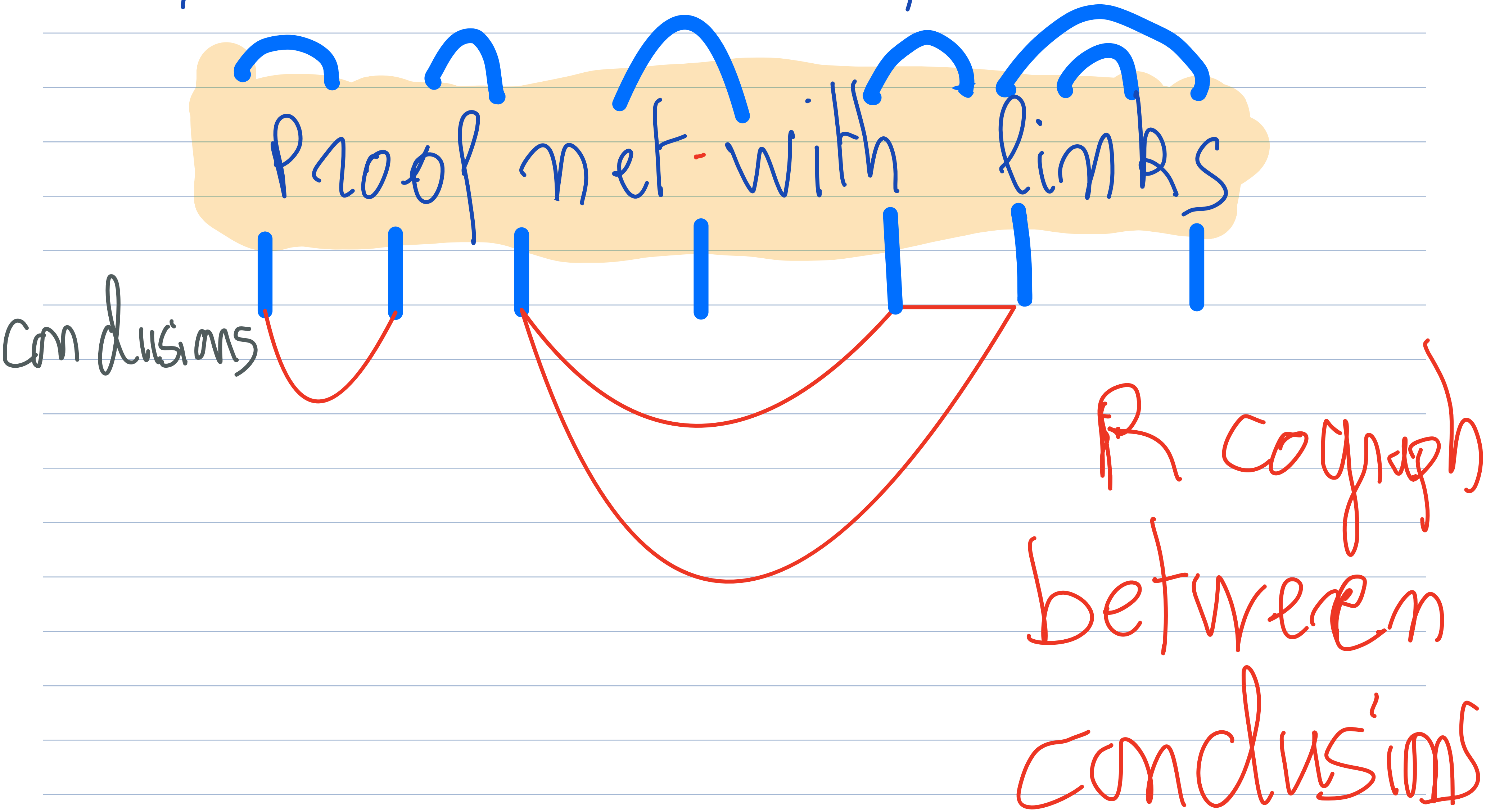
$\Downarrow$

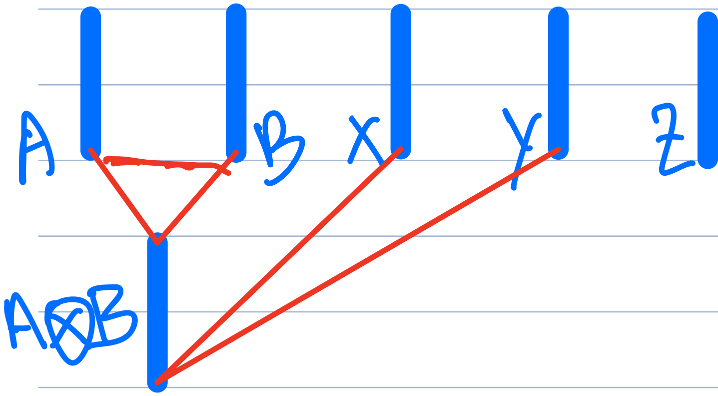
B-bridge

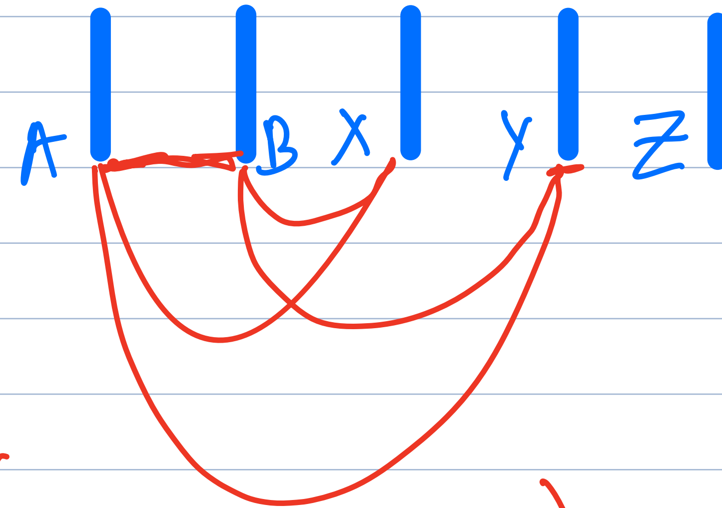
$\Downarrow$

sequentialisation.

without Mix add: an ae-path between any two vertices fold / unfold


Proof net with links
conclusions
R cograph between conclusions



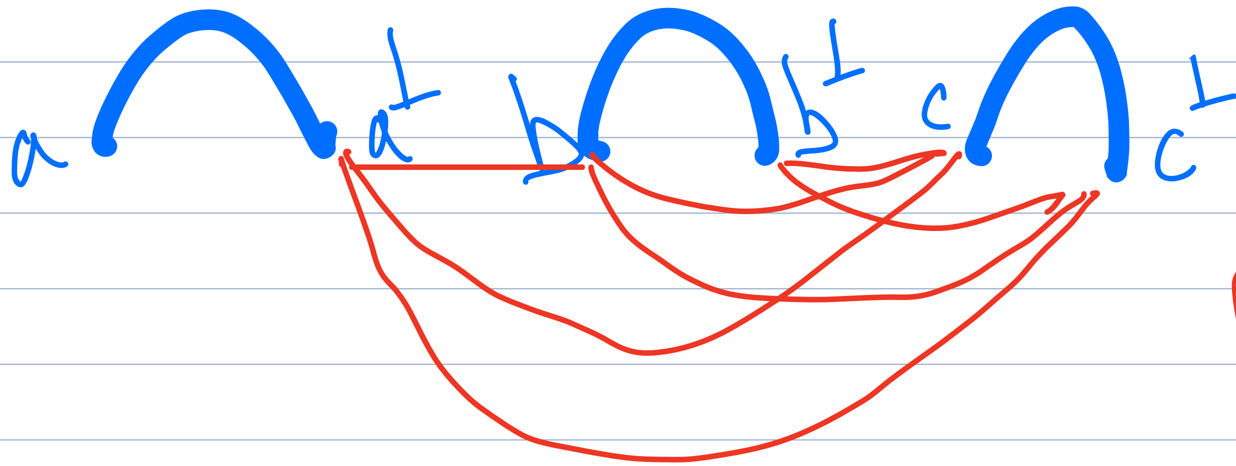


$$((A \otimes B) \otimes (X ⅋ Y)) ⅋ Z \to (A \otimes B \otimes [X ⅋ Y]) ⅋ Z$$

de-cycle may appear →

every de-cycle contains a chord preserved in both direction

Handsome proofnets



axioms

R: formula = cograph

$a \parr ((a^\perp \otimes b) \parr b^\perp) \otimes (c \parr c^\perp))$

every ae cycle antimoadnal

Red cograph + perfect matching

fold → fold → fold →

proof net with links

(detail: axiom axiom)

In proofnet with bicoloured links every ea-cycle contains a chord IFF no ea cycle at all

→ B bridge sequentialisation

all sub cographs by rewriting:

① ~~$(a ⅋ b) \otimes (c ⅋ d) \rightsquigarrow (a \otimes c) ⅋ (b \otimes d)$~~ WRONG

② $(a ⅋ b) \otimes c \rightsquigarrow a ⅋ (b \otimes c)$

③ $a \otimes c \rightsquigarrow a ⅋ c$ (Mix)

---

rewriting modulo:

- associativity $\otimes$ ⅋
- commutativity $\otimes$ ⅋

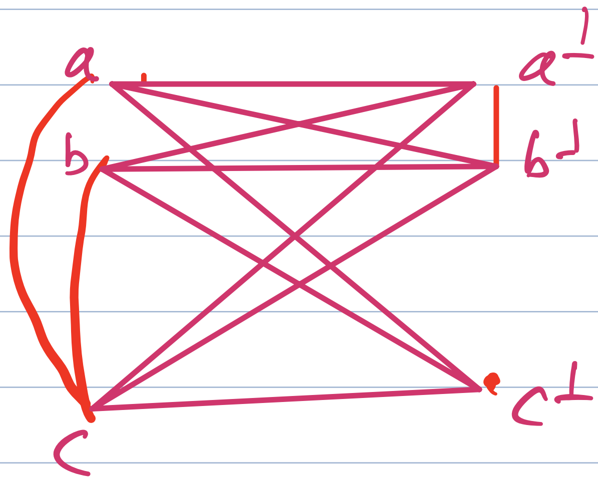
From $\bigotimes_i (\alpha_i \parr \alpha_i^\perp)$ (axiom)

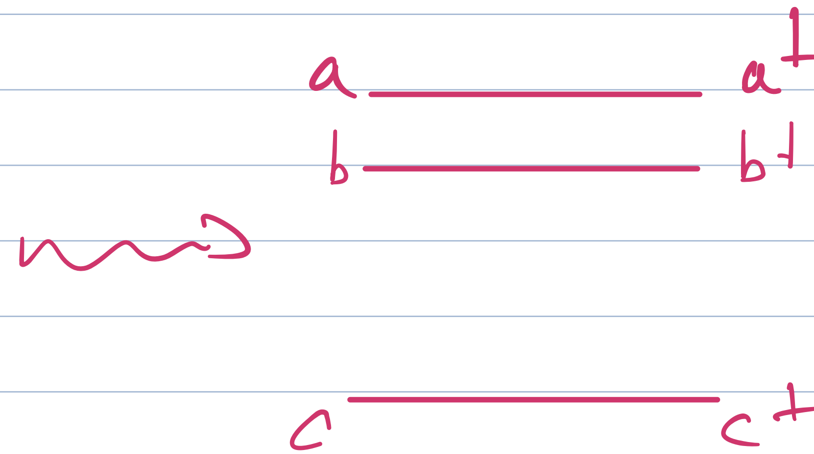
- $(a \parr b) \otimes c \multimap a \parr (b \otimes c)$
  yields all MLL theorems
  adding
- $a \otimes c \multimap a \parr c$ (Mix)
  yield all MLL+Mix theorem

---

rewriting is deep inference viewpoint

cut elimination comes from rewriting (2)





$(a \parr b) \otimes c \qquad (a^\perp \otimes b^\perp) \parr c^\perp$

cut

$a \quad b \quad c \qquad a^\perp \quad b^\perp \quad c^\perp$

cut

cut

cut

POMSET LOGIC

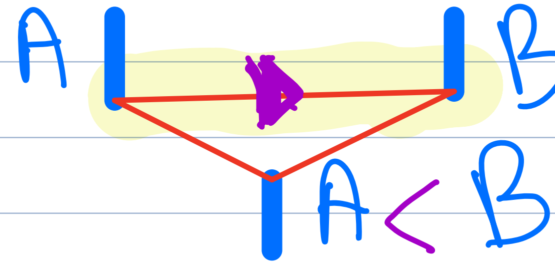
from coherence semantics

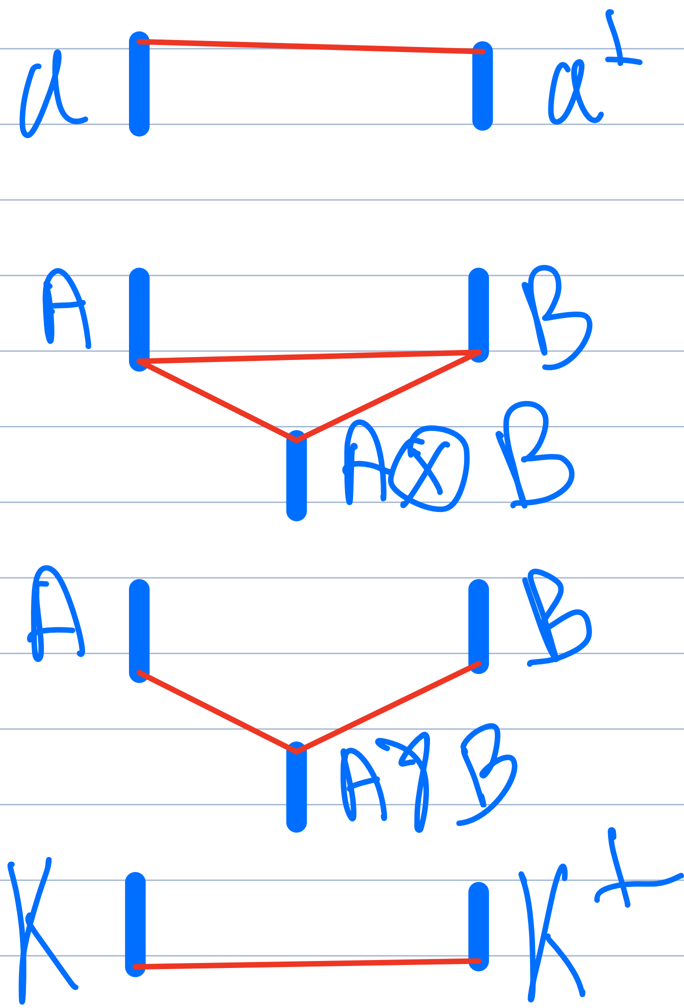
- associative
- non commutative
- self dual $(A < B)^\perp \equiv A^\perp < B^\perp$

(Mix)

$$\begin{tikzcd}
 & A < B \arrow[dr, dash] & \\
A \otimes B \arrow[ur, dash] \arrow[dr, dash] & & A \mathbin{⅋} B \\
 & B < A \arrow[ur, dash] &
\end{tikzcd}$$

$< \text{linR}$





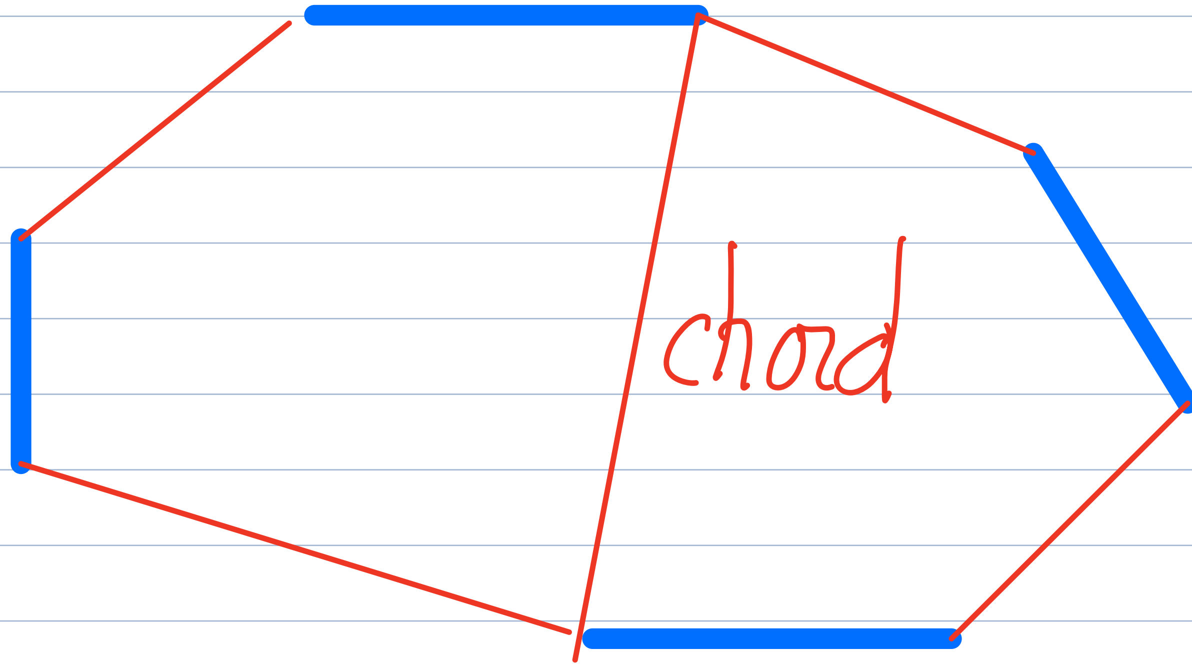
	criterion:
axiom	
⊗	no
⅋	ae-circuit
cut	stable by cut-elim.

Fold / unfold: same transformation

Criterion for handsome PN:
every ae circuit (directed)
contains a directed chord

Not necessarily a splitting B-edge
when there is no ae circuit

ae-cycle



perfect matching

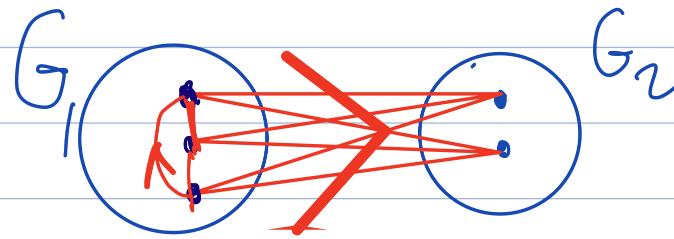
other edges

Directed cographs, inductively

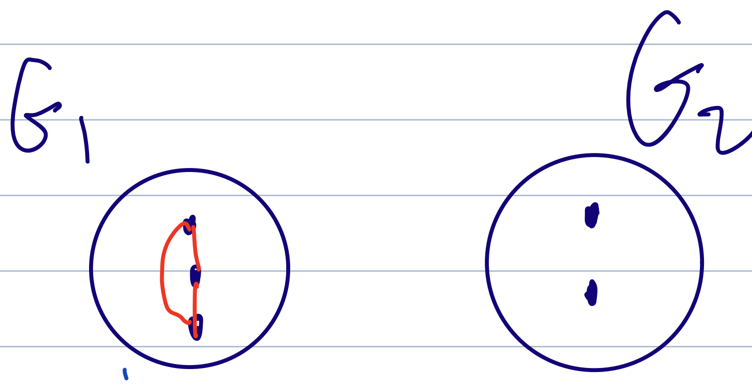
$<$ operation

$\overline{=}$

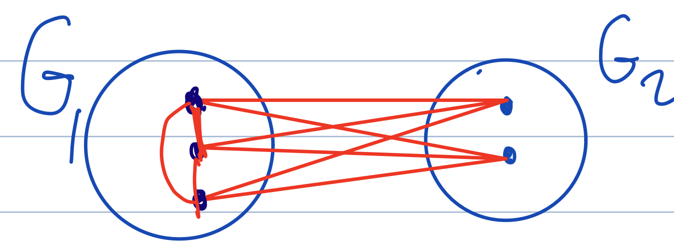
directed $\otimes$



$G_1 < G_2$



$G_1 \parr G_2$

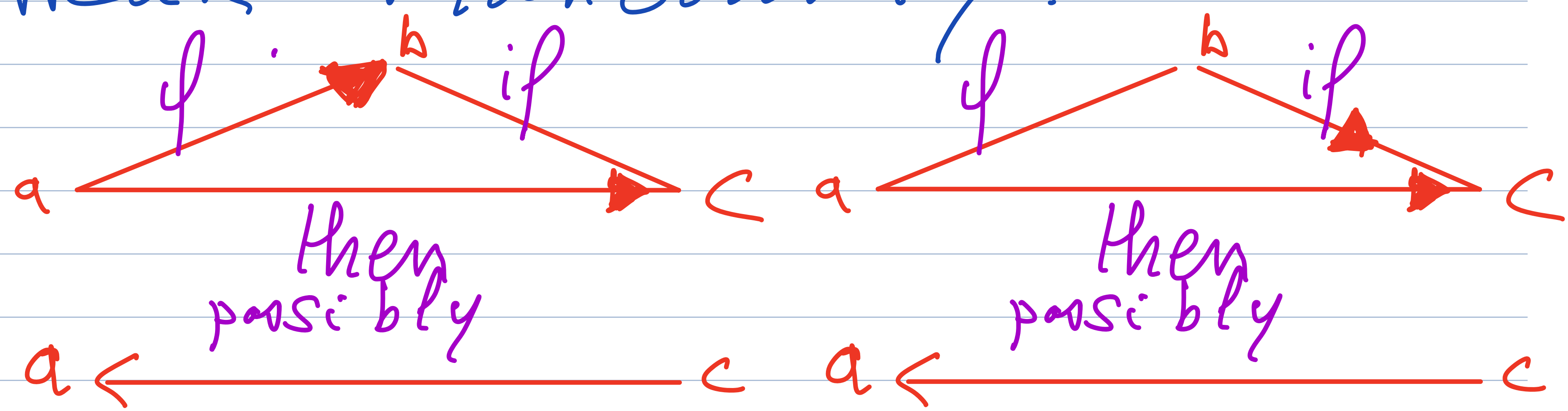


$G_1 \otimes G_2$

Directed cographs characterisation

viewed as $\longleftrightarrow$

- the symmetric part $\otimes G$ is a cograph
  P_4 free :
- the directed part $< G$ is an SP order
  Hasse diagram is N-free
- weak transitivity:



Rewriting/inclusion of directed cographs
add:

- $(a<b)\otimes(c<d) \multimap (a\otimes c)<(b\otimes d)$
  and its variant with 3 or 2 terms
- $(a \parr b) < (c \parr d) \multimap (a<c) \parr (b<d)$
  and it's variant with 3 or 2 terms

They all preserve correction.

(S)BV deep inference

fold / unfold idem

cut elimination

coherence semantics

a PN is correct iff
its interpretation is a clique
in the coherence space
associated with the conclusions.

no sequent calculus :-(

rewriting modulo algebraic properties

(S)BV $\neq$ Pomset logic

Guglielmi
Strassburger

↓

no faithfull semantic

interpretation of π is a clique in the associated coherens space

IFF

Proof Net is Correct

Slavnov (2019)

complete sequent calculus

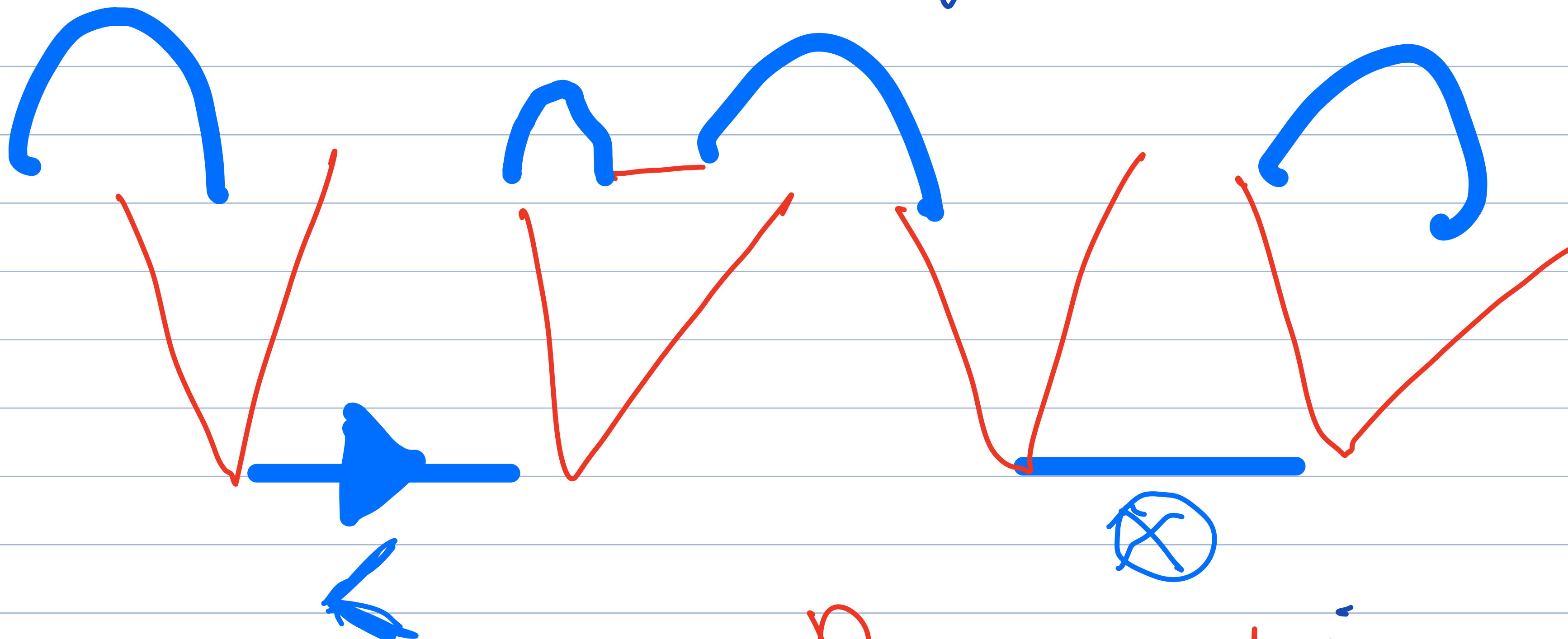
but if n - conclusions

$\lfloor \frac{n}{2} \rfloor$ binary relations:

$R_p \; (c_1 .. c_p)(c'_1 \; c'_p)$

p disjoint paths $\begin{cases} c_1 \text{———} c'_p \\ \quad \vdots \\ c_p \text{———} c'_p \end{cases}$

Other edge bicoloured graphs
intermediate. aggregates / handsome PN



R : $K_{1,p}$

R symmetric

An interesting idea from Slavnov

$$(A \lhd B)^\perp = A^\perp \lhd B^\perp$$

is perhaps a degenerat case

$$(A \otimes B)^\perp = A^\perp \parr B^\perp$$

In 1992 I was desperatly
lookin for a sequent calculus
for pomset Logic ......
fortunately I defended my PhD without
Now I'm emeritus
I can think about it.