A contribution to the history of the Curry-Howard correspondence: Prawitz's 1968/1970 homomorphism

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Why not one name more?

The Brouwer - Heyting - Kolmogorov - Schönfinkel - Curry - Meredith - Kleene - Feys - Gödel - Läuchli - Kreisel - Tait - Lawvere - Howard - de Bruijn - Scott - Martin-Löf - Girard - Reynolds - Stenlund - Constable - Coquand - Huet - ... isomorphism might be a more appropriate name, still not including all the contributors. [Sørensen & Urzyczyn, 2006, p. viii]

The first aim of this talk is showing that one more name should be added to the list, i.e., that of Dag Prawitz.

(It is actually quite unfair that Prawitz is not mentioned...)

Prawitz's normalization

Prawitz's normalisation theory for Natural Deduction is relevant to the history of the Curry-Howard isomorphism already because, today, we conceive of the isomorphism as one between Natural Deduction and typed λ-calculus. This is due to Martin-Löf. Howard used (a variant of) Sequent Calculus.

But this is *not* the kind of contribution by Prawitz that I focus on here...

Prawitz's 1970 paper

In 1970, Prawitz published a paper titled *Constructive semantics* in which he defines an early example of a mapping between Natural Deduction derivations and a language of construction terms very similar to typed λ-calculus. The mapping is to a large extent structural in the same sense as above.

But the paper was ready (at least in the main lines) already in 1968, since it appeared in the *Proceedings of the 1st Scandinavian Logic Symposium*, held in Åbo in September that year.

This means that Prawitz's work predates by one year Howard's *The formulae-as-types notion of construction*, whose manuscript started circulating in 1969 (and was published only in 1980).

Prawitz's preliminary observations

Prawitz's aim "is to study the question of the semantical interpretation of constructive logic" [Prawitz 1970].

The focus is on the interpretation of the logical constants, for which Prawitz refers to BHK semantics. `IL` is looked at as an *object-level* embedding of constructivist intuitions which are lost in classical logic.

Elementary calculi

An *elementary calculus* $\mathfrak{S}$ is determined by individual, function and predicate constants of a first-order language $\mathscr{L}$, plus a set of *rules* $A_1, ..., A_n \Rightarrow B$ with A_i, B atomic.

An *instance* of a rule is obtained by uniformly replacing free parameters by closed terms over $\mathfrak{S}$ in the rule. A *derivation of* A *in* $\mathfrak{S}$ is a sequence $\langle B_1, ..., B_n \rangle$ such that $B_n = A$ and, for all $i \leq n$, B_i is an atomic sentence over $\mathfrak{S}$ and, for some $B_j, ..., B_h$ $(j, h \leq i)$, $B_j, ..., B_h \Rightarrow B_i$ is an instance of a rule in $\mathfrak{S}$.

Types and construction objects

The *type grammar* is:

- T$_1$) 0 is a type
- T$_2$) τ_1, τ_2 are types $\Rightarrow \langle \tau_1, \tau_2 \rangle$ and $\tau_1(\tau_2)$ are types.

The *construction objects of type* τ *over* $\mathfrak{S}$ are:

- O$_1$) terms and formulas of $\mathscr{L}$, as well as derivations in $\mathfrak{S}$ are construction objects of type 0 over $\mathfrak{S}$
- O$_2$) $k_i :^{\mathfrak{S}} \tau_i \ (i = 1, 2) \Rightarrow \langle k_1, k_2 \rangle :^{\mathfrak{S}} \langle \tau_1, \tau_2 \rangle$
- O$_3$) k is an effective procedure from objects of type τ_1 over any $\mathfrak{S}' \supseteq \mathfrak{S}$ to objects of type τ_2 over $\mathfrak{S}' \Rightarrow k :^{\mathfrak{S}} \tau_1(\tau_2)$.

Types of formulas are easy, e.g., $A_1 \rightarrow A_2 : \tau_1(\tau_2)$, $\forall x A : 0(\tau)$, $\exists x A : \langle 0, \tau \rangle$.

Constructions of formulas

k is a *construction of a sentence* A *over* $\mathfrak{S}$, written $\mathtt{C}(k, A, \mathfrak{S})$, iff

At) $A \in \mathtt{ATOM}_{\mathscr{L}} \Rightarrow k$ is a derivation of A in $\mathfrak{S}$

$\vee$) $A = A_1 \vee A_2 \Rightarrow k = \langle\langle k_1, k_2\rangle, A_i\rangle$ and, either $i = 1$ and $\mathtt{C}(k_1, A_1, \mathfrak{S})$ with k_2 arbitrary construction object of type τ over $\mathfrak{S}$ and $A_2 : \tau$, or $i = 2$ and $\mathtt{C}(k_2, A_2, \mathfrak{S})$ with k_1 arbitrary construction object of type τ over $\mathfrak{S}$ and $A_1 : \tau$

$\rightarrow$) $A = A_1 \rightarrow A_2 \Rightarrow k :^{\mathfrak{S}} \tau_1(\tau_2)$, $A : \tau_1(\tau_2)$ and, for every $\mathfrak{S}' \supseteq \mathfrak{S}$ and every k', if $\mathtt{C}(k', A_1, \mathfrak{S}')$, then $\mathtt{C}(k(k'), A_2, \mathfrak{S}')$.

Constructive and intuitionistic truth

A is *constructively true over* $\mathfrak{S}$ iff there is k such that $\mathtt{C}(k, A, \mathfrak{S})$, and it is *constructively valid* iff, for every $\mathfrak{S}$, A is constructively true over $\mathfrak{S}$.

Intuitionistic truth over $\mathfrak{S}$ and *intuitionistic validity* are defined in the same way, except that we refer to *consistent* elementary calculi, i.e., elementary calculi where $\perp$ is not provable.

Towards construction terms and soundness

Prawitz aims now to prove that "minimal (intuitionistic) logic is sound with respect to constructive (intuitionistic) validity" [Prawitz 1970].

Officially, one must prove that, if $\Gamma \vdash B$, then B is true over every $\mathfrak{S}$ provided the elements of Γ are. This is not proved by associating directly to every B a Γ-depending construction over every elementary calculus, but by letting the required construction objects be denoted by "terms that we shall [...] call *construction terms*" [Prawitz 1970].

What Prawitz wants to prove is actually a stronger result, i.e., that every *derivation* in `ML` or `IL` denotes a construction of the required kind.

Construction terms

The *language of construction terms* has the following dictionary:

- denumerably many typed variables $\alpha^\tau, \beta^\tau, ...$ for every type τ
- constants $\mathrm{p}, \mathrm{p}_i \ (i = 1, 2), \lambda, \{\ \}(\ ), \mathsf{c}, \mathsf{S}$
- the terms and formulas of the underlying first-order language.

The set of the *typed construction terms* is:

K$_1$) $\alpha^\tau, \beta^\tau : \tau$

K$_2$) $t : 0$ for every t and $A : 0$ for every A

K$_3$) $T_i : \tau_i \ (i = 1, 2) \Rightarrow \mathrm{p}(T_1, T_2) : \langle \tau_1, \tau_2 \rangle$

K$_4$) $T : \langle \tau_1, \tau_2 \rangle \Rightarrow \mathrm{p}_i(T) : \tau_i \ (i = 1, 2)$

K$_5$) $T : \tau_2 \Rightarrow \lambda \alpha^{\tau_1}(T) : \tau_1(\tau_2)$

K$_6$) $T : \tau_1(\tau_2)$ and $U : \tau_1 \Rightarrow \{T\}(U) : \tau_2$

K$_7$) $T_1, T_2 : \tau$ and $U_1, U_2 : 0 \Rightarrow \mathsf{c}(T_1, T_2, U_1, U_2) : \tau$

K$_8$) $T^n = \langle T_1, ..., T_n \rangle, U^n = \langle U_1, ..., U_n \rangle, T_i, U_i, Z : 0 \Rightarrow \mathsf{S}(T^n, U^n, Z) : 0.$

Evaluations

A $\mathfrak{S}$-*assignment* is a function v from typed variables to construction objects over $\mathfrak{S}$ such that $v(\alpha^\tau) :^{\mathfrak{S}} \tau$. The $\mathfrak{S}'_k$ of v over α^τ is an assignment which is the same as v except for assigning $k :^{\mathfrak{S}'} \tau$ to α^τ.

To any $\mathfrak{S}$-assignment v, we associate a *valuation* Φ_v:

E$_1$) $\Phi_v(t) = t$ for every t and $\Phi_v(A) = A$ for every A

E$_2$) $\Phi_v(\alpha^\tau) = v(\alpha^\tau)$

E$_3$) $\Phi_v(\lambda\alpha^\tau(U)) = f$ such that $f(k) = \Phi_{v^*}(U)$, where v^* is the $\mathfrak{S}'_k$-variant of v over α^τ

E$_4$) $\Phi_v(\mathsf{c}(T_1, T_2, U_1, U_2)) = \begin{cases} \Phi_v(T_1) & \text{if } \Phi_v(U_1) = \Phi_v(U_2) \\ \Phi_v(T_2) & \text{otherwise} \end{cases}$

E$_5$) $\Phi_v(\mathsf{S}(T^n, U^n, Z)) = (\Phi_v(Z))(\Phi_v(U_1), ..., \Phi_v(U_n)/\Phi_v(T_1), ..., \Phi_v(T_n))$.

Appropriateness

I abstract from free individual parameters. To deal with them, one introduces a suitable sequence of variables of type 0, and one replaces free parameters with closed terms which variables of type 0 are mapped to by assignments.

T is *appropriate for* $\Gamma \Rightarrow B$, with $\Gamma = \langle A_1, ..., A_n \rangle$ and $A_i : \tau_i$, iff

A$_1$) $FV(T) \subseteq \{\alpha^{\tau_1}, ..., \alpha^{\tau_n}\}$

A$_2$) for every $\mathfrak{S}$, every $k_1, ..., k_n$ construction objects over $\mathfrak{S}$, it holds that: if

$$\mathsf{C}(k_i, A_i, \mathfrak{S})$$

then

$$\mathsf{C}(\Phi_v(T), B, \mathfrak{S})$$

where $v(\alpha^{\tau_i}) = k_i$.

Intuitionistic appropriateness holds over consistent calculi.

Preliminaries on soundness

> *To each provable sentence A we shall find a construction term T [...] such that the value of T is a construction of A (with respect to any calculus over which A is a formula). [Prawitz 1970]*

Prawitz does not say that his proof amounts to defining a homomorphism, which I call ph. In [Prawitz 1971], however, he *does* acknowledge that.

The theorem is proved by induction on the derivation.

Overall assessment

Prawitz's homomorphism is the same as the Curry-Howard isomorphism on the $\{\to, \forall, \bot\}$-fragment, i.e., limited to λ-abstraction and application (except for being a homomorphism, see below).

On the full language, the mapping works fine only from natural deduction to the language of construction terms. There are however problems when trying to map construction terms back to natural deduction. The proof that to any term appropriate for $\Gamma \Rightarrow B$ there corresponds a derivation of $\Gamma \vdash B$ is not so smooth either.

However, the main insights of the Curry-Howard isomorphism are there.

Chronology

The paper is drawn from a talk at the *First Scandinavian Logic Symposium*, held in Turku/Åbo in September 1968.

Howard's manuscript started circulating only somewhere in 1969.

Prawitz mentions in a footnote a similar approach by Howard [Prawitz 1970]. But he confirmed in private communication that he got Howard's work while he was already writing the 1968 talk down, and was not influenced by it.

Prawitz says that a refinement of the typing was suggested to him by Zucker, presumably during Prawitz's stay in the US in 1964.

The content of the paper was anticipated by a talk at a seminar in Stockholm in 1963, and at one in UCLA in 1964.

There are occasional references to the fact that Prawitz had independently something similar to the Curry-Howard isomorphism [Prawitz 1971, 1981; Sundholm 1998].

The Curry-Howard isomorphism is a three-layered correspondence:

1. Formulas = Types
2. Derivations = Programs
3. Normalization = Computation

Prawitz lacks 1. Formulas *are not types*. They are typed, and they are even construction objects and construction terms.

Another direct consequence is that Prawitz's mapping is an homomorphism. For example, the one step proofs for $A \to A$ and $B \to B$ are mapped onto α-equivalent construction terms (might even be the same construction term).

Dependent types

The most important consequence is the absence of dependent types (which were already in de Bruijn 1968).

For example, the type of $P(t_1, ..., t_n)$ is always 0, *for every* $t_1, ..., t_n$.

Another example. Consider

$$\dfrac{\dfrac{[P(a)]}{P(a) \to P(a)} \to_I}{\forall x(P(x) \to P(x))} \forall_I$$

This is mapped onto $\lambda\beta^0.\lambda\alpha^0.\alpha^0$. So, β^0 is *vacuously* bound. This is gotten rid of only at the level of valuation (namely, by considering suitable variants of assignments to β^0).

This putting the burden on valuations is not coincidental though.

Meta-theoretical perspective

Prawitz does not look at his morphism as an identity between derivations and programs (let alone formulas and types). The language of construction terms is a *meta-language* for proving soundness of `ML` or `IL`.

This is so also in Howard, but for him λ-terms are constructions *as such*. In other words, he follows the pattern:

$$\text{derivations} \Longrightarrow \lambda\text{-terms} = \text{constructions}$$

whereas for Prawitz it is

$$\text{derivations} \Longrightarrow \text{construction terms} \Longrightarrow \text{constructions}$$

(similar to the Gödel translation).

Thus, Howard's approach lends itself better to the current way of conceiving of the Curry-Howard isomorphism.

THANK YOU