Ordinals, Cardinals and Bar Recursion in Classical Realizability

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Plan of this talk

- Introduction to classical realizability
- Ordinals and choice principles in classical realizability models
- Cardinals in classical realizability models

Realizability

Realizability: interprets mathematical theories in some model of computation in order to extract the computational content of mathematical proofs.

We establish a correspondence between formulas and programs in a way that is compatible with the rules of deduction.

From the BHK interpretation to realizability

Intuitionism

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Kleene 1945

A proof is a *computable* function that exhibits a witnesses of the proven statement, hence a general recursive function.

From Curry Howard to Forcing

Logical system

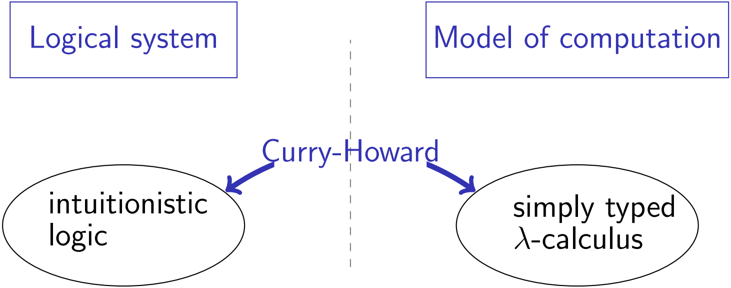
Model of computation

Kleene's Realisability

Heyting Arithmetic

recursive functions

From Curry Howard to Forcing



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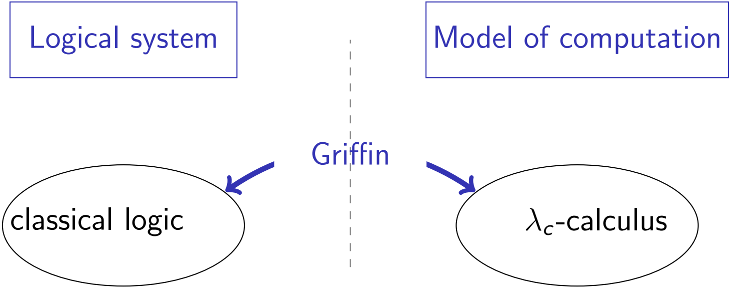

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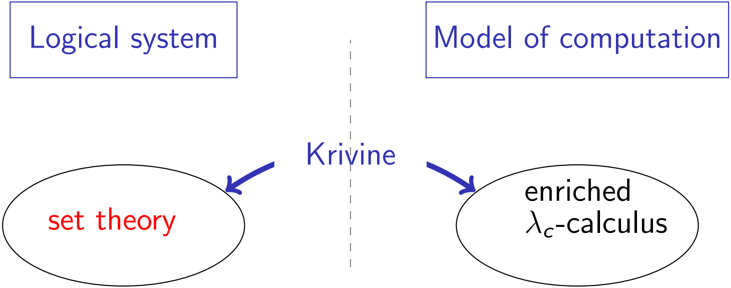

Forcing (Boolen Valued Models) in a nutshell

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We consider a complete Boolean algebra $(\mathbb{B}, 0, 1, \wedge_{\mathbb{B}} \vee_{\mathbb{B}}, \neg_{\mathbb{B}})$, and we "evaluate" the truth each formula of ZF using the elements of $\mathbb{B}$:

$$\varphi \mapsto ||\varphi|| \in \mathbb{B}$$

$$||\varphi \wedge \psi|| = ||\varphi|| \wedge_{\mathbb{B}} ||\psi||$$
$$||\varphi \vee \psi|| = ||\varphi|| \vee_{\mathbb{B}} ||\psi||$$
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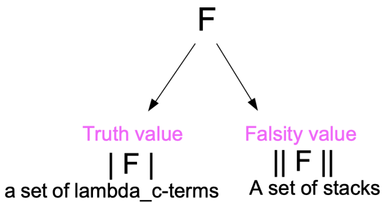
Given an ultrafilter U on $\mathbb{B}$, the $\text{Theory} := \{\varphi \ : \ \exists p \in U(p \Vdash \varphi)\}$ forms a coherent classical theory which contains ZF(C) (if we do the process starting from a model of ZF(C), the *ground model*)

Classical realizability in a nutshell



A program $p \Vdash F$ ("p realizes F") when it is in the truth value of F, that is when p is 'orthogonal' to every stack π in the falsity value of F,

meaning that the process $p * \pi \in \perp\!\!\!\perp$, where $\perp\!\!\!\perp$ is the so-called 'Pole'.

Classical realizability in a nutshell

Then we choose some privileged λ_c-terms that we call *coherent realizers*/*proof-like terms*, here denoted $\mathcal{PL}$.

$$T = \{\varphi \mid \exists\theta \in \mathcal{PL}(\theta \Vdash |\varphi|)\}$$

Classical realizability in a nutshell

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We show that T forms a coherent theory, which is closed by classical natural deduction and under the right constraints it will contain the axioms we want to realize (e.g. ZF); a realizability model is a model of such a theory.

Realizability algebras

The main ingredients are :

- Λ a set of programs (λ_c-terms$+$ ...)
- Π a set of stacks
- $\mathcal{PL}$ a set of realizers/proof-like terms
- $\prec_K$ the execution, a pre-order on processes $t * \pi$ (where t is a program and π is a stack)
- $\perp\!\!\!\perp$ the pole a set of processes $\prec_K$-upward closed (it defines the "orthogonal processes")

Terms and stacks

$\Lambda^{\text{open}}_{A,B}$ (λ_c-**terms**) :

$$
\begin{array}{rcll}
t,u & ::= & x & \text{(variable; we choose a countable set of variables} \\
 & | & tu & \text{(application)} \\
 & | & \lambda x.t & \text{(abstraction; } x \text{ is a variable and } t \text{ is a } \lambda_c\text{-term)} \\
 & | & \mathit{cc} & \text{(call-with-current-continuation)} \\
 & | & k_\pi & \text{(continuation constants; } \pi \text{ is a stack)} \\
 & | & \xi_\alpha & \text{(special terms; } \alpha \in A\text{)}
\end{array}
$$

$\Pi_{A,B}$ (**Stacks**) :

$$
\begin{array}{rcll}
\pi & ::= & \omega_\beta & \text{(stack bottoms; } \beta \in B\text{)} \\
 & | & t \bullet \pi & \text{(} t \text{ is a closed } \lambda_c\text{-term and } \pi \text{ is a stack)}
\end{array}
$$

The execution

$\prec_K$ is the smallest preorder on the set of processes such that

$$\begin{array}{rcll} tu * \pi & \succ_K & t * u \bullet \pi & \text{(push)} \\ \lambda x.t * u \bullet \pi & \succ_K & t[x := u] * \pi & \text{(grab)} \\ cc * t \bullet \pi & \succ_K & t * k_\pi \bullet \pi & \text{(save)} \\ k_{\pi'} * t \bullet \pi & \succ_K & t * \pi' & \text{(restore)} \end{array}$$

The pole

$\perp\!\!\!\perp$ the pole a set of processes $\prec_K$-upward closed.

The algebra is coherent if for every proof-like term t there is $\pi \in \Pi$ such that $t \star \pi$ is NOT in the pole.

Non extensional set theory

In order to realize the axioms of set theory, we work with a *non-extensional* version of ZF, called ZF_ε, with...

two membership relations:

- $\in$ the usual extensional one
- ε a (strict) non-extensional one

$$\forall x \forall y [x \in y \iff \exists z \;\varepsilon\; y (x \simeq z)]$$
$$\forall x \forall y [x \subseteq y \iff \forall z \;\varepsilon\; x (z \in y)]$$

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... and two equality relations

- $\simeq$ the usual extensional one
- $=$ Leibniz identity

Truth and Falsity Values for ZF_ε

We define $|\varphi|$ and $||\varphi||$ simultaneously and by induction on φ in the language of realizability.

- no stack is in $||\top||$, every stack is in $||\bot||$
- $||a \not\varepsilon b|| = \{\pi \text{ stack };\ (a, \pi) \in b\}$
- $||a \subseteq b|| = \{\xi \mathbin{.} \pi;\ \exists c (c, \pi) \in a \text{ and } \xi \Vdash c \notin b\}$
- $||a \notin b|| = \{\xi \mathbin{.} \xi' \mathbin{.} \pi;\ \exists c (c, \pi) \in b \text{ and } \xi \Vdash a \subseteq c \text{ and } \xi' \Vdash c \subseteq a\}$
- $||\varphi \Rightarrow \psi|| = \{\xi \mathbin{.} \pi;\ \xi \Vdash \varphi \text{ and } \pi \in ||\psi||\}$
- $||\forall x \varphi|| = \bigcup_a ||\varphi[a/x]||$

$\xi \in |\varphi| \iff \forall \pi \in ||\varphi|| (\xi \star \pi \in \perp\!\!\!\perp)$

$\xi \Vdash \varphi$ means $\xi \in |\varphi|$

Realizability models for set theory

call-cc realizes Peirce's law $(((A \Rightarrow B) \Rightarrow A) \Rightarrow A)$

ZF_ε is a conservative extension of ZF. The realizability theory contains the axioms of ZF_ε (provided ZF is consistent).

Any model of such a theory yields a structure (actually many) in the language of ZF_ε denoted $\mathcal{N}_\varepsilon$ or $\mathcal{N}$, and a structure (actually many) in the language of ZF, denoted $\mathcal{N}_\in$.

Realizability models for set theory

Thus the axioms of ZF are realizable.

Exemple:

- (Krivine 2003) the Pairing axiom is realized by $\lambda x.((xI)I)$,
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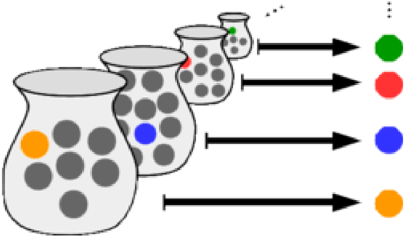
- (Krivine 2003) the Axiom of Foundation is realized by any fixed point combinator
- (F. + Geoffroy + Matthews 2024) Several large cardinals axioms can be realized by pure lambda-terms

What is the computational content of the axiom of choice?



Realizing dependent choice

Krivine 2003-2016

Dependent Choice can be realized using quote/a clock and or the bar recursion.

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$$quote * t.s.\pi \succ t * \underline{n_s}.\pi$$

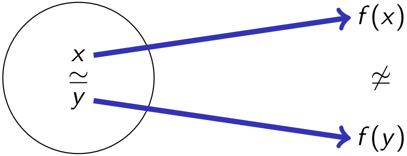
where $s \mapsto n_s$ is some fixed enumeration of Λ in order-type ω.
Here *quote* works as a comparator of terms.

Non extensional axiom of choice

Using quote, one can realize a non extensional version of AC, called NEAC

NEAC: existence of a *non-extensional* choice function, i.e.

$$\begin{array}{lcl} x = y & \Rightarrow & f(x) = f(y), \text{ but} \\ x \simeq y & \not\Rightarrow & f(x) \simeq f(y) \end{array}$$



Fragments of AC and ordinal representatives

Suppose that we have a set $a \simeq \gamma$ such that

$$\forall x, y \varepsilon a (x = y \iff x \simeq y),$$

then AC_γ holds.

Because NEAC gives as a non-extensional choice function f which is extensional on a, hence on γ.

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Hence, realizing fragments of AC depends on our ability to represent ordinals with sets that behave well with the two equalities.

Ordinals in Classical realizability

Ordinals in set theory

Over ZF, equivalent definitions of ordinals:

1. An ordinal is a transitive set which is well-ordered by $\in$.
2. An ordinal is a transitive set a satisfying trichotomy. That is, for any $x, y, \in a$, precisely one of the following hold: $x \in y$, $y \in x$, or $x = y$.
3. An ordinal is a transitive set of transitive sets.

Ordinals in ZF_ε

In ZF_ε these notions are **not** equivalent.

1. an ε-transitive set which is ε-well-ordered.
2. an ε-TOD : an ε-transitive set a satisfying ε-trichotomy.
3. an ε-ordinal: an ε-transitive set of ε-transitive sets

The first two are equivalent and stronger.

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The first two are equivalent and stronger. Moreover,
if α is an ε-TOD, then for every $x, y \varepsilon \alpha$, $x \simeq y$ if and only if $x = y$.
On the other hand, non trivial realizability models have only set many ε-TODs (F. + Matthews 2024). The notion of ε-ordinal is enough since it implies $\in$-ordinal.

Hat ordinals - F. + Geoffroy 2021

We fix terms $\{\underline{\alpha}\}_{\alpha<\kappa}$ (e.g. if $\kappa = \omega$, take Church numerals).

For every $\alpha \leq \kappa$,

$$\hat{\alpha} := \{(\hat{\beta}, \underline{\beta} \bullet \pi) | \pi \in \Pi, \beta < \alpha\}$$

Proposition

For every $\alpha \leq \kappa$, we can realize that $\hat{\alpha}$ is an ε-ordinal.
Moreover, if α is a successor ordinal (in the ground model), then $\hat{\alpha}$ is a successor ordinal (in the realizability model),
and if α is a limit ordinal, then $\hat{\alpha}$ is a limit ordinal.

Recursive ordinals - F. + Matthews 2024

The Gimel function:

$$\gimel x = x \times \Pi$$

The Reish function (recursive):

$$ר(x) = \{(ר(y), \pi);\ y \in x, \pi \in \Pi\}$$

In particular,

$$ר(\alpha) = \{(ר(\beta), \pi);\ \beta < \alpha, \pi \in \Pi\}$$

In non trivial realizability models (i.e. non-forcing models), $ר(2)$ has more than two ε-elements (4, infinitely many...).

Recursive ordinals

F. + Matthews 2024

- $\daleth\alpha$ and $\hat{\alpha}$ are always ε-ordinals, hence $\in$-ordinals.
- $\daleth\omega \simeq \hat{\omega} \simeq \omega$
- $\daleth ORD$ is extensionally equivalent to the class of ordinals in the realizability model

Trichotomy Operator

Definition

A trichotomy operator for a sequence of terms $(\underline{\xi})_{\xi<|\mathcal{A}|}$ is a term $\chi \in \Lambda_c$ such that for any $\alpha, \beta < \kappa$ and any $t, u \in \Lambda_c, \pi \in \Pi$ we have

$$(\chi) \quad \chi\underline{\alpha}\, fg\underline{\beta} \star \pi \succ_\chi \begin{cases} (f)\underline{\beta} \star \pi & \text{if } \beta < \alpha; \\ g \star \pi & \text{if } \beta \geq \alpha \end{cases}$$

Given a term f, the trichotomy operator is used to represent sequences $(f\underline{0}, ..., f\underline{\alpha}, \underline{0}, ...\underline{0}...)$, since for instance $\chi\underline{\alpha+1}f\underline{0}\underline{\beta}$ reduces to $f\underline{\beta}$ if $\beta \leq \alpha$ and to $\underline{0}$ otherwise.

epsilonTODs

Let κ be the size of the realizability algebra.

- (F. + Geoffroy 2021) if χ is a proof-like term, then every $\hat{\alpha}$ is an ε-TOD

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- (F. + Matthews 2024) if the realizability model is non trivial (i.e. $\daleth 2 \neq 2$), then ε-TODs are bounded by $\daleth(\kappa^+)$. (hence they extensionally form a set)

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- (F. + Matthews 2024) if the realizability model is non trivial (i.e. $\daleth 2 \neq 2$), then ε-TODs are bounded by $\daleth(\kappa^+)$. (hence they extensionally form a set)
- recursive ordinals are usually not ε-TODs

Cardinals in classical realizability models

Preserving cardinals

In Forcing and in classical realizability, cardinals may 'collapse' (ex. κ may be an uncountable cardinal in the ground model, but $\hat{\kappa}$ may become a countable ordinal in the realizability model).

- cardinals strictly greater than the size of the forcing are preserved
- a forcing with the κ-chain condition preserves all cardinals $\geq \kappa$
- a forcing with the κ-closure property preserves all cardinals $\leq \kappa$

Preserving cardinals

- F. + Matthews 2024 : cardinals strictly greater than the size of the realizability algebra are preserved
- F. + Geoffroy 2021 / F. Matthews 2024 : a κ-chain condition for realizability algebras to preserve $\hat{\kappa}$ and $\daleth\kappa$
- Krivine 2016 : an $\aleph_1$-closure property for realizability algebras
- F. + Furlan 2025 : a κ-closure property for realizability algebras to preserve all cardinals $\leq \hat{\kappa}$

Preserving cardinals with the δ-chain condition

Definition

A boolean algebra has the δ-chain condition if every antichain has size $< \delta$.

Definition (F. + Matthews 2024)

A realizability algebra satisfies the uniform δ-chain condition if there exists a proof-like term $p \in \mathcal{PL}$ such that for every $A \subseteq \Lambda$ of cardinality at least δ, for every $t \in \Lambda$ and $\pi \in \Pi$:
if for every $a \neq b$ in A $(t \star a \bullet b \bullet \pi \in ⫫)$,
then there is $a \in A$ such that $p \star t \bullet a \bullet \pi \in ⫫$.

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if for every $a \neq b$ in A $(t \star a \bullet b \bullet \pi \in \perp\!\!\!\perp)$,
then there is $a \in A$ such that $p \star t \bullet a \bullet \pi \in \perp\!\!\!\perp$.

For boolean algebras (forcings) this is equivalent to the set theoretic δ-chain condition.

Preserving cardinals with the δ-chain condition

F. + Matthews 2024

Suppose that a realizability algebra satisfies the uniform δ-chain condition for some regular cardinal δ, then $\daleth\delta$ is an $\in$-cardinal in the realizability model

F. + Matthews 2024

For every $\delta > |\Lambda|$, the realizability algebra satisfies the uniform δ-chain condition.

Definition

A poset $(\mathbb{P}, \leq)$ has the κ-closure if for every (limit) $\gamma < \kappa$, for every decreasing sequence $(p_\xi)_{\xi<\gamma}$ of elements of $\mathbb{P}$, there is $p \in \mathbb{P}$ such that $p \leq p_\xi$ for every ξ

κ-closure for realizability algebras - F. + Furlan 2025

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Definition

Given a sequence $(u_\xi)_{\xi<\gamma}$ for some ordinal γ, we say that $t \in \Lambda_c$ accumulates a sequence $(u_\xi)_{\xi<\gamma}$ if for any $\xi < \gamma$ $(t)\underline{\xi} \succ u_\xi$.

κ-closure for realizability algebras - F. + Furlan 2025

Definition

A realizability algebra $\mathcal{A}$ is κ-fully closed for some $\kappa \in Ord$ if and only if for every limit ordinal $\gamma < \kappa$, and for every sequence $(u_\xi)_{\xi<\gamma} \subseteq \Lambda_c$ the following properties are satisfied:

1. there is a term u that accumulates $(u_\xi)_{\xi<\gamma}$;
2. for every term v, if $v\{x := u\} \Vdash \perp$, then there is $\delta < \gamma$ such that for every accumulator w of the sequence $(u_\xi)_{\xi<\delta}$ we have $v\{x := w\} \Vdash \perp$.

In the case of a boolean algebra $\mathbb{B}$, this coincides with the set theoretic κ-closure for $\mathbb{B}^+$.

A non trivial realizability algebra with the κ-closure

A non trivial realizability algebra satisfying the κ-closure can be obtained adding oracles, namely assuming that for every sequence of terms $(u_\xi)_{\xi<\gamma}$ with γ limit, there is a term denoted $\bigwedge_{\xi<\gamma} u_\xi$ such that for every $\beta < \gamma$

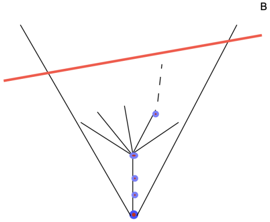
$$(\bigwedge_{\xi<\gamma} u_\xi)\underline{\beta} \succ u_\beta$$

Bar induction

- A *bar* for a tree $T(x)$ over $A^{<\omega}$ is a predicate $B(x)$ such that $B(x) \Rightarrow T(x)$ and for every ω-branch b for $T(x)$, there exists $n < \omega$ such that $B(b \restriction n)$.
- A bar $B(x)$ for T is *inductive* if the following condition holds: $\forall s \in A^{<\omega}(T(s) \wedge \forall a \in A(T(s \frown a) \Rightarrow B(s \frown a)) \rightarrow B(s))$;

Bar Induction

For every set A, given a tree T on $A^{<\omega}$ and a bar B for T, if the bar is inductive, then the empty sequence satisfies the bar.



A transfinite bar induction

Given a limit ordinal γ,

- A *bar* for a tree $T(x)$ over $A^{<\gamma}$ is a predicate $B(x)$ such that $B(x) \Rightarrow T(x)$ and for every γ-branch b for $T(x)$, there exists $\delta < \gamma$ such that $B(b \restriction \delta)$.
- A bar $B(x)$ for T is *inductive* if the following conditions hold:
  1. $\forall s \in A^{<\gamma}(T(s) \wedge \forall a \in A(T(s \frown a) \Rightarrow B(s \frown a)) \rightarrow B(s))$;
  2. for all increasing sequences $(s_\alpha)_{\alpha<\delta}$ with $\delta < \gamma$, $B(\bigcup_{\alpha<\delta} s_\alpha) \Rightarrow \exists \alpha < \delta \ B(s_\alpha)$.

Transfinite Bar-Induction

For a limit ordinal γ, states that for any set A, given a tree T on $A^{<\gamma}$ and a bar B for T, if the tree is closed by $\sqsubseteq$-increasing sequences of length $< \gamma$ and the bar is inductive, then the empty sequence satisfies the bar.

The Bar Recursion- Krivine's style

For Y any fix-point operator,

$$BR := \lambda gu.(Y)\lambda hif.(u)(\chi if)(g)\lambda z.(h(\underline{s})i)(\chi)ifz$$

Here χ is a trichotomy operator, but not necessarily a comparator of all terms.

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Here χ is a trichotomy operator, but not necessarily a comparator of all terms.
For any $g, u, f \in \Lambda_c, n$, the bar-induction operator reduces as

$$BRgu\underline{n}f \succ (u)(\chi \underline{n} f)(g)\lambda z(BRgu(\underline{s})\underline{n})(\chi)\underline{n}fz.$$

$$BRgu\underline{n}f \succ (u)(Restr[f, \underline{n}])(g)Ind[g, u, f, \underline{n}].$$

A generalized/transfinite Bar Recursion

For Y any fix-point operator,

$$BR := \lambda gu.(Y)\lambda hif.(u)(\chi if)(g)\lambda z.(h(\underline{s})i)(\chi)ifz$$

For any $g, u, f \in \Lambda_c, \alpha < \kappa$, the bar-induction operator reduces as

$$BRgu\underline{\alpha}f \succ (u)(\chi\underline{\alpha}f)(g)\lambda z(BRgu(\underline{s})\underline{\alpha})(\chi)\underline{\alpha}fz.$$

$$BRgu\underline{\alpha}f \succ (u)(Restr[f, \underline{\alpha}])(g)Ind[g, u, f, \underline{\alpha}].$$

Preserving cardinals with the bar induction

Theorem (F. + Furlan 2025)

Suppose that $\mathcal{A}$ is a realizability algebra that satisfies the κ-full closure and has "an inductive system of terms" (roughly a way to encode sequences of terms with other terms), then using the transfinite bar recursion BR we can realize $AC_{<\hat{\kappa}}$ and for every cardinal $\mu \leq \kappa$, we can realize that $\hat{\mu}$ is a cardinal (i.e. cardinalities are preserved up to $\hat{\kappa}$).

Future work

- Use transfinite bar recursion to realize $DC_{\hat{\kappa}}$ or $ZL_{\hat{\kappa}}$
- Use transfinite bar recursion to realize generalized versions of CH.
- Work with a proper class of hat ordinals $\{\underline{\alpha}\}_{\alpha \in Ord}$ to realize the full axiom of choice with the transfinite bar recursion or a comparator.

Thank you