No Uniform Interpolation for Linear Logic

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Interpolation

Definition: Craig's Interpolation

An **interpolant** for two formulas A and B with $A \vdash B$ is a formula C s.t.:

- $\mathrm{Voc}(C) \subseteq \mathrm{Voc}(A) \cap \mathrm{Voc}(B)$
- $A \vdash C$ is provable
- $C \vdash B$ is provable

$$\forall A, \forall B, \exists C, \ldots$$

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Definition: Uniform Interpolation

A **uniform interpolant** for a formula A and an atom X is a formula C s.t.:

- $\mathrm{Voc}(C) \subseteq \mathrm{Voc}(A) \setminus \{X\}$
- $A \vdash C$ is provable
- for all formulas B such that $X \notin \mathrm{Voc}(B)$ and $A \vdash B$ is provable, $C \vdash B$ is also provable

Interpolation

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A <u>**uniform**</u> **interpolant** for a formula A and an atom X is a formula C s.t.:

- $\text{Voc}(C) \subseteq \text{Voc}(A) \setminus \{X\}$
- $A \vdash C$ is provable $\boxed{\approx \exists \textit{ right introduction}}$
- for all formulas B such that $X \notin \text{Voc}(B)$ and $A \vdash B$ is provable, $C \vdash B$ is also provable $\boxed{\approx \exists \textit{ left introduction}}$

Some Logics with(out) Uniform Interpolation

Classical Logic ✓

Easy

$$A \quad \rightsquigarrow \quad A[\top/X] \vee A[\bot/X]$$

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Intuitionistic Logic ✓ **[Pitts 1992]**

Syntactic approach, contraction rule problematic

$$(X \to Y) \to Z \quad \rightsquigarrow \quad ((Y \to Z) \to Y) \to Z$$

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Modal Logic S4 ✗ **[Ghilardi and Zawadowski 1995]**

Explicit **counter-example** with a game model

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(I)MALL ✓ **[Alizadeh, Derakhshan, and Ono 2014]**

Same **syntactic** method as for Intuitionistic Logic

$X^+ \otimes (X^+ \multimap Y^+) \quad \rightsquigarrow \quad Y^+$

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(I)LL ✗ **[this talk]**

Explicit **counter-example** with a formula

Linear Logic (1/2)

Formulas

$$
\begin{array}{lll}
A, B ::= & \mid X^- \mid X^+ & \textit{atom} \\
& \mid A \parr B \mid A \otimes B \mid \bot \mid 1 & \textit{multiplicative} \\
& \mid A \oplus B \mid A \,\&\, B \mid 0 \mid \top & \textit{additive} \\
& \mid ?A \mid !A & \textit{exponential}
\end{array}
$$

Involutive Negation

$$(X^-)^\perp = X^+ \quad (X^+)^\perp = X^-$$

$$(A \parr B)^\perp = A^\perp \otimes B^\perp \quad (A \otimes B)^\perp = A^\perp \parr B^\perp \quad \bot^\perp = 1 \quad 1^\perp = \bot$$

$$(A \oplus B)^\perp = A^\perp \,\&\, B^\perp \quad (A \,\&\, B)^\perp = A^\perp \oplus B^\perp \quad 0^\perp = \top \quad \top^\perp = 0$$

$$(?A)^\perp = !A^\perp \quad (!A)^\perp = ?A^\perp$$

Linear Arrow

$$A \multimap B = A^\perp \parr B$$

Linear Logic (2/2)

Rules

Cut-Elimination

$$\frac{}{\vdash X^-, X^+}\ (ax) \qquad \frac{\vdash A^\perp, \Gamma \quad \vdash A, \Delta}{\vdash \Gamma, \Delta}\ (cut)$$

$$\frac{\vdash A, B, \Gamma}{\vdash A \parr B, \Gamma}\ (\parr) \qquad \frac{\vdash A, \Gamma \quad \vdash B, \Delta}{\vdash A \otimes B, \Gamma, \Delta}\ (\otimes) \qquad \frac{\vdash \Gamma}{\vdash \bot, \Gamma}\ (\bot) \qquad \frac{}{\vdash 1}\ (1)$$

$$\frac{\vdash A, \Gamma \quad \vdash B, \Gamma}{\vdash A \,\&\, B, \Gamma}\ (\&) \qquad \frac{\vdash A, \Gamma}{\vdash A \oplus B, \Gamma}\ (\oplus_1) \qquad \frac{\vdash B, \Gamma}{\vdash A \oplus B, \Gamma}\ (\oplus_2) \qquad \frac{}{\vdash \top, \Gamma}\ (\top)$$

$$\frac{\vdash A, \Gamma}{\vdash ?A, \Gamma}\ (?d) \qquad \frac{\vdash ?A, ?A, \Gamma}{\vdash ?A, \Gamma}\ (?c) \qquad \frac{\vdash \Gamma}{\vdash ?A, \Gamma}\ (?w) \qquad \frac{\vdash A, ?\Gamma}{\vdash !A, ?\Gamma}\ (!)$$

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The Counter-Example

Goal: Exhibit formulas A and $(B_n)_{n \in \mathbb{N}}$ s.t.:

1. $X \in \mathrm{Voc}\,(A)$ and $X \notin \mathrm{Voc}\,(B_n)$
2. $\vdash A^{\perp}, B_n$ is provable
3. for all formulas C such that $X \notin \mathrm{Voc}\,(C)$,
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Solution:

$$A = X^{+} \otimes {!}\left(X^{+} \multimap (X^{+} \otimes Y^{+})\right) \otimes (X^{+} \multimap Y^{+})$$

$$B_n = \underbrace{Y^{+} \otimes \ldots \otimes Y^{+}}_{n+1 \text{ times}}$$

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Idea:

2. Feed successively X^+ to n copies of $!\,(X^+ \multimap (X^+ \otimes Y^+))$ to get n times Y^+, then consume X^+ through $(X^+ \multimap Y^+)$.

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Idea:

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3. *"C has no useful $!$ on Y^+"* since $!(X^+ \multimap (X^+ \otimes Y^+))$ from A cannot really be used above a promotion. Hence *"$C^\perp$ has no useful $?$ on Y^-"* so it has $(n+1)$ different Y^- to link to the $(n+1)$ Y^+ of B_n.

Case in point: candidate $!Y^+ \otimes Y^+$ but $\vdash A^\perp, !Y^+ \otimes Y^+$ **not** provable!

The easy part: $\boxed{1}$ and $\boxed{2}$

Lemma $\boxed{1}$

$X \in \mathrm{Voc}(A)$ and $X \notin \mathrm{Voc}(B_n)$ for all $n \in \mathbb{N}$

Proof.

Trivial. $\square$

Lemma $\boxed{2}$

$\vdash A^\perp, B_n$ is provable (intuitionistically) for all $n \in \mathbb{N}$

Proof.

Provide the proof trees by an easy induction on n.

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{(ax)}
\UnaryInfC{$\vdash X^-, X^+$}
\AxiomC{}
\RightLabel{(ax)}
\UnaryInfC{$\vdash X^-, X^+$}
\AxiomC{}
\RightLabel{(ax)}
\UnaryInfC{$\vdash Y^-, Y^+$}
\AxiomC{}
\RightLabel{(ax)}
\UnaryInfC{$\vdash Y^-, Y^+$}
\RightLabel{$(\otimes)$}
\BinaryInfC{$\vdash Y^-, Y^-, Y^+ \otimes Y^+$}
\RightLabel{$(\otimes)$}
\BinaryInfC{$\vdash X^-, Y^-, X^+ \otimes Y^-, Y^+ \otimes Y^+$}
\RightLabel{$(\parr)$}
\UnaryInfC{$\vdash X^- \parr Y^-, X^+ \otimes Y^-, Y^+ \otimes Y^+$}
\RightLabel{$(\otimes)$}
\BinaryInfC{$\vdash X^-, X^+ \otimes (X^- \parr Y^-), X^+ \otimes Y^-, Y^+ \otimes Y^+$}
\RightLabel{$(?_d)$}
\UnaryInfC{$\vdash X^-, ?(X^+ \otimes (X^- \parr Y^-)), X^+ \otimes Y^-, Y^+ \otimes Y^+$}
\RightLabel{$(\parr)$}
\UnaryInfC{$\vdash X^-, ?(X^+ \otimes (X^- \parr Y^-)) \parr (X^+ \otimes Y^-), Y^+ \otimes Y^+$}
\RightLabel{$(\parr)$}
\UnaryInfC{$\vdash X^- \parr ?(X^+ \otimes (X^- \parr Y^-)) \parr (X^+ \otimes Y^-), Y^+ \otimes Y^+$}
\end{prooftree}
$$

$\square$

The hard part: 3

Idea: *"C has no useful ! on Y^+"* for $!(X^+ \multimap (X^+ \otimes Y^+))$ from A is useless without some X^+. Hence, $C^\perp$ contains a Y^- for each Y^+ in B_n.

Lemma: *"C has no useful ! on Y^+"*

*Let C be such that $X \notin \mathrm{Voc}(C)$ and $\vdash A^\perp, C$ is provable. There is a cut-free proof of $\vdash A^\perp, C$ where !-**rules have no X in their conclusions**.*

Proof.

Not complicated: do some substitutions $X \mapsto 0$, some compositions by *cut*, and use the sub-formula property. □

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Lemma 3

Let C be a formula with $X \notin \mathrm{Voc}(C)$. If $\vdash A^\perp, C$ and $\vdash C^\perp, B_n$ are provable, then C contains (at least) $n+1$ occurrences of Y^+.

Proof?

Yep. Indeed.

How to **propagate** the data from the previous result to *ax*-rules on B_n?

Geometry Of Interaction in poor man's proof-nets

Definition: GOI projection graph $\mathcal{G}_\pi$ of a proof π of $\vdash \Gamma$

Undirected edge-bicoloured graph defined as:

Example

π

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{
        \dfrac{}{\vdash X^-, X^+}\,(ax)
      }{\vdash ?X^-, X^+}\,(?_d)
    }{\vdash ?X^-, !X^+}\,(!)
    \quad
    \dfrac{
      \dfrac{}{\vdash X^-, X^+}\,(ax)
    }{\vdash ?X^-, X^+}\,(?_d)
  }{\vdash ?X^-, !X^+ \,\&\, X^+}\,(\&)
  \quad
  \dfrac{
    \dfrac{
      \dfrac{
        \dfrac{
          \dfrac{}{\vdash X^-, X^+}\,(ax)
        }{\vdash ?X^-, X^+}\,(?_d)
        \quad
        \dfrac{
          \dfrac{}{\vdash X^-, X^+}\,(ax)
        }{\vdash ?X^-, X^+}\,(?_d)
      }{\vdash ?X^-, ?X^-, X^+ \otimes X^+}\,(\otimes)
    }{\vdash ?X^-, X^+ \otimes X^+}\,(?_c)
  }{\vdash ?X^- \oplus X^-, X^+ \otimes X^+}\,(\oplus_1)
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$$

$\mathcal{G}_\pi$

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\dfrac{
\dfrac{
\dfrac{\dfrac{\dfrac{}{\vdash X^-, X^+}\,{\scriptstyle (ax)}}{\vdash ?X^-, X^+}\,{\scriptstyle (?_d)}}{\vdash ?X^-, !X^+}\,{\scriptstyle (!)}
\quad
\dfrac{\dfrac{}{\vdash X^-, X^+}\,{\scriptstyle (ax)}}{\vdash ?X^-, X^+}\,{\scriptstyle (?_d)}
}{\vdash ?X^-, !X^+ \,\&\, X^+}\,{\scriptstyle (\&)}
\quad
\dfrac{\dfrac{\dfrac{
\dfrac{\dfrac{}{\vdash X^-, X^+}\,{\scriptstyle (ax)}}{\vdash ?X^-, X^+}\,{\scriptstyle (?_d)}
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\dfrac{\dfrac{}{\vdash X^-, X^+}\,{\scriptstyle (ax)}}{\vdash ?X^-, X^+}\,{\scriptstyle (?_d)}
}{\vdash ?X^-, ?X^-, X^+ \otimes X^+}\,{\scriptstyle (\otimes)}}{\vdash ?X^-, X^+ \otimes X^+}\,{\scriptstyle (?_c)}}{\vdash ?X^- \oplus X^-, X^+ \otimes X^+}\,{\scriptstyle (\oplus_1)}
}{\vdash ?X^-, X^+ \otimes X^+}\,{\scriptstyle (cut)}
$$

$\mathcal{G}_\pi$

$$
\underbrace{X^-}_{?X^-}\qquad
\underbrace{X^+ \qquad X^+}_{\textit{cut}\text{-formula } !X^+ \,\&\, X^+}\qquad
\underbrace{X^- \qquad X^-}_{\textit{cut}\text{-formula } ?X^- \oplus X^-}\qquad
\underbrace{X^+ \qquad X^+}_{X^+ \otimes X^+}
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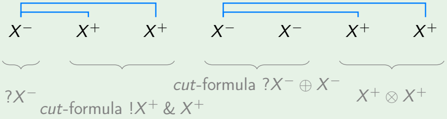

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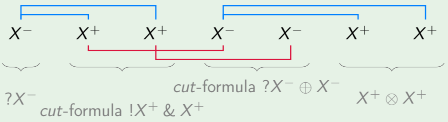

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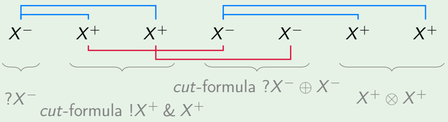

Several *ax*-edges on an occurrence: either due to a $?_c$-rule or to a $\&$-rule

Alternating persistent paths

Alternating path: consecutive edges are coloured differently

Lemma

Assume $\pi \longrightarrow \rho$*. If there is an alternating path between two vertices in* $\mathcal{G}_\rho$*, then there also is an alternating path between them in* $\mathcal{G}_\pi$*.*

Proof.

Simple inspection on the kind of the cut-elimination step. □

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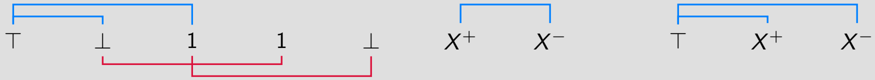
Simple inspection on the kind of the cut-elimination step. □

$\top$ Warning $\perp$

Adding units and $\top$-edges for $\top$-rules is a **bad** idea: the above lemma fails!

$$\dfrac{\dfrac{}{\vdash \top, \perp \parr 1}\,(\top) \qquad \dfrac{\dfrac{}{\vdash 1}\,(1) \qquad \dfrac{\dfrac{}{\vdash X^+, X^-}\,(ax)}{\vdash \perp, X^+, X^-}\,(\perp)}{\vdash 1 \otimes \perp, X^+, X^-}\,(\otimes)}{\vdash \top, X^+, X^-}\,(cut) \qquad \longrightarrow \qquad \dfrac{}{\vdash \top, X^+, X^-}\,(\top)$$



Back to Lemma 3 $(A^\perp = X^- \parr ?(X^+ \otimes (X^- \parr Y^-)) \parr (X^+ \otimes Y^-))$

Lemma 3

Let C be a formula with $X \notin \mathrm{Voc}(C)$. If $\vdash A^\perp, C$ and $\vdash C^\perp, B_n$ are provable, then C contains (at least) $n+1$ occurrences of Y^+.

Proof.

1. Cut + normalise the two proofs: *ax*-rules with each Y^+ and a Y^-

$$\underbrace{Y_1^-}_{A^\perp} \qquad \underbrace{Y^+ \quad \cdots \quad Y_\alpha^+ \quad \cdots \quad Y^+}_{B_n}$$

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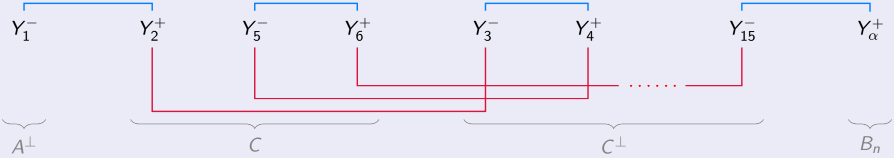

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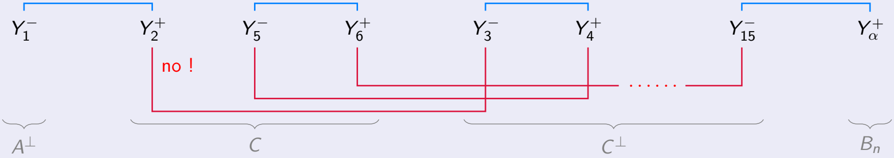
1. Cut + normalise the two proofs: *ax*-rules with each Y^{+} and a Y^{-}
2. These are **alternating paths**: we track each back *before normalising*
3. Lemma *"C has no useful ! on Y"*; **propagation along the path**



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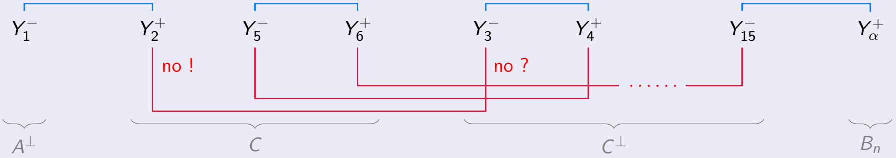
Let C be a formula with $X \notin \mathrm{Voc}(C)$. If $\vdash A^\perp, C$ and $\vdash C^\perp, B_n$ are provable, then C contains (at least) $n+1$ occurrences of Y^+.

Proof.

1. Cut + normalise the two proofs: *ax*-rules with each Y^+ and a Y^-
2. These are **alternating paths**: we track each back *before normalising*
3. Lemma *"C has no useful $!$ on Y"*; **propagation along the path**



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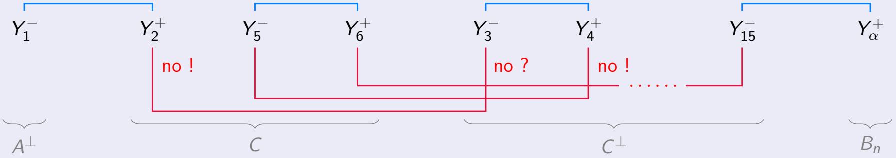

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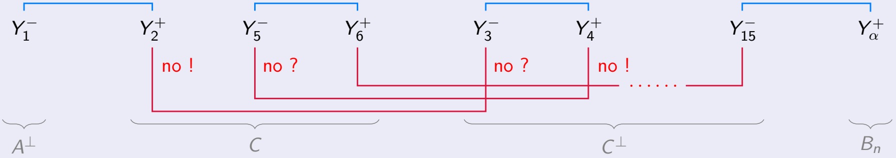

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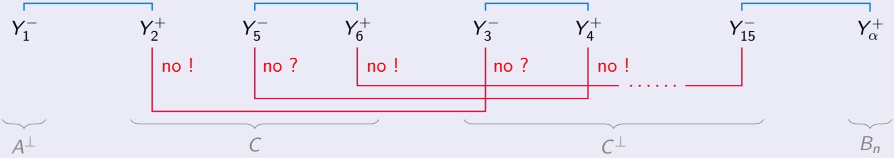

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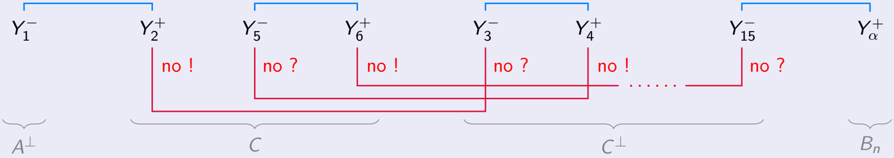

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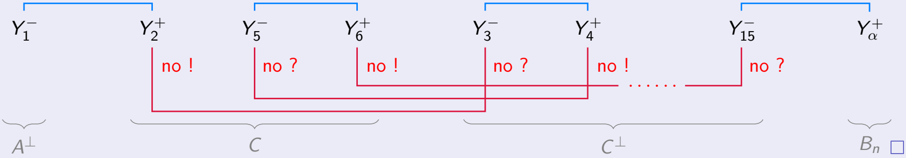
2. These are **alternating paths**: we track each back *before normalising*
3. Lemma "*C has no useful $!$ on Y*"; **propagation along the path**
4. Do everything *from the start* in a 0-slice: take the proof and delete one sub-tree of every &-rule *that is not above a $!$-rule*



□

& Warning !

We can adapt everything done before with 0-**slices** but not with **slices**.

Conclusion

Proposition

Linear Logic does not have uniform interpolation.

Proof.

No interpolant for $A = X^{+} \otimes !(X^{+} \multimap (X^{+} \otimes Y^{+})) \otimes (X^{+} \multimap Y^{+})$ $\square$

Conclusion

Proposition

(Intuitionistic) Linear Logic does not have uniform interpolation.

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No interpolant for $A = X^+ \otimes !(X^+ \multimap (X^+ \otimes Y^+)) \otimes (X^+ \multimap Y^+)$
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Observations

- Why a counter-example for **Linear Logic** and not for *Classical Logic*, or *Intuitionistic Logic*, or *MALL*?
  $\longrightarrow$ A with a sub-formula that can be contracted, and one that cannot

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  $\longrightarrow$ Proof of Lemma 3 without GOI projection graphs?

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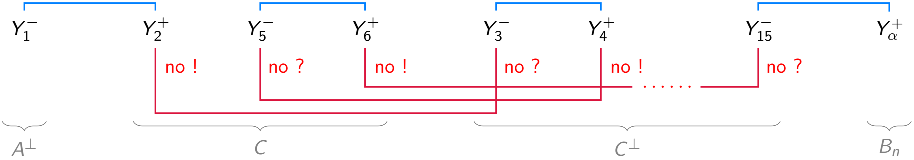
- Problem about **provability** but *GOI* and *cut-elimination* very useful
  $\longrightarrow$ Proof of Lemma 3 without GOI projection graphs?

Perspectives

- **Decision problem:** given some A, does it have a uniform interpolant?
- Proof simple enough to be **formalized** in Rocq?

Thank you for your attention!



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Ghilardi, Silvio and Marek Zawadowski (June 1995). "Undefinability of propositional quantifiers in the modal system S4". In: *Studia Logica* 55.2, pp. 259–271. DOI: 10.1007/BF01061237.

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Back-up: Proof of "C *has no useful* $!$ *on* Y^+"

Lemma: *"C has no useful $!$ on Y^+"*

*Let C be such that $X \notin \mathrm{Voc}(C)$ and $\vdash A^\perp, C$ is provable. There is a cut-free proof of $\vdash A^\perp, C$ where $!$-**rules have no X in their conclusions**.*

Proof.

Let π be a cut-free derivation of $\vdash A^\perp, C$ *with a minimal number of $!$-rules having $?(X^+ \otimes (X^- \parr X^-))$ in their conclusion sequents.* We prove there are none. By the sub-formula property, any given $!$-rule is of the shape

$$\dfrac{\begin{array}{c}\rho\\ \vdash (?(X^+ \otimes (X^- \parr Y^-)))^i, D, ?\Delta\end{array}}{\vdash (?(X^+ \otimes (X^- \parr Y^-)))^i, !D, ?\Delta}\ (!)$$

where $!D, ?\Delta$ are sub-formulas of C. If $i \geq 1$, then we contradict the minimality of π with π where we replace this sub-tree with a normal form of

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash \top, \mathbf{0} \otimes Y^+}\ (\top)}{\vdash \top \parr (\mathbf{0} \otimes Y^+)}\ (\parr)}{\vdash !\left(\top \parr (\mathbf{0} \otimes Y^+)\right)}\ (!) \qquad \dfrac{\begin{array}{c}\rho[0/X]\\ \vdash (?(\mathbf{0} \otimes (\top \parr Y^-)))^i, D, ?\Delta\end{array}}{\vdash ?(\mathbf{0} \otimes (\top \parr Y^-)), D, ?\Delta}\ (?_c)}{\dfrac{\vdash D, ?\Delta}{\vdash !D, ?\Delta}\ (!)}\ (cut)}{\vdash (?(X^+ \otimes (X^- \parr Y^-)))^i, !D, ?\Delta}\ (?_w)$$

□

Back-up: 0-slice = slice at depth 0

Definition: 0-slice

A 0-**slice** of a proof π is the result of deleting one of the two sub-trees of each &-rule of π <u>*that is not above a !-rule*</u>. *Some* &-rules become unary:

$$\frac{\vdash A, \Gamma}{\vdash A \,\&\, B, \Gamma}\ (\&_1) \qquad \frac{\vdash B, \Gamma}{\vdash A \,\&\, B, \Gamma}\ (\&_2)$$

Example

$$\dfrac{\dfrac{}{\vdash \top, !(1\&1)}\,(\top) \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\vdash 1}\,(1) \quad \dfrac{}{\vdash 1}\,(1)}{\vdash 1\&1}\,(\&)}{\vdash !(1\&1)}\,(!)}{\vdash \bot, !(1\&1)}\,(\bot)}{\vdash \top\&\bot, !(1\&1)}\,(\&)$$

$$\rightsquigarrow \quad \dfrac{\dfrac{}{\vdash \top, !(1\&1)}\,(\top)}{\vdash \top\&\bot, !(1\&1)}\,(\&_1)$$

$$\rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash 1}\,(1) \quad \dfrac{}{\vdash 1}\,(1)}{\vdash 1\&1}\,(\&)}{\vdash !(1\&1)}\,(!)}{\vdash \bot, !(1\&1)}\,(\bot)}{\vdash \top\&\bot, !(1\&1)}\,(\&_2)$$

Back-up: 0-slice and cut-elimination

Cut-elimination extends to 0-slices in the natural way:

$$\dfrac{\dfrac{\vdash \Gamma, A}{\vdash \Gamma, A \& B}\ (\&_1) \qquad \dfrac{\vdash A^\perp, \Delta}{\vdash A^\perp \oplus B^\perp, \Delta}\ (\oplus_1)}{\vdash \Gamma, \Delta}\ (cut) \qquad \longrightarrow \qquad \dfrac{\vdash \Gamma, A \qquad \vdash A^\perp, \Delta}{\vdash \Gamma, \Delta}\ (cut)$$

$$\dfrac{\dfrac{\vdash \Gamma, A}{\vdash \Gamma, A \& B}\ (\&_1) \qquad \dfrac{\vdash B^\perp, \Delta}{\vdash A^\perp \oplus B^\perp, \Delta}\ (\oplus_2)}{\vdash \Gamma, \Delta}\ (cut) \qquad \not\longrightarrow$$

Lemma

If $\pi \longrightarrow \rho$, then for each 0-slice ρ' of ρ, there exists a 0-slice π' of π such that $\pi' \longrightarrow \rho'$ or $\pi' = \rho'$.

Proof.

Simple inspection on the kind of the cut-elimination step. □

& Warning !

The above result holds for 0-**slices** but not for **slices**: failure for a $?_c - !$ key cut-elimination case. See for instance $?(A \oplus B) \simeq ?A \parr ?B$.

Back-up: Affine Logic

Same as Linear Logic except:

$$\cancel{\dfrac{\vdash \Gamma}{\vdash ?A, \Gamma}\,(?w)} \quad \rightsquigarrow \quad \dfrac{\vdash \Gamma}{\vdash A, \Gamma}\,(w)$$

$$\cancel{\dfrac{\dfrac{\vdash \Gamma}{\vdash ?A, \Gamma}\,(?w) \qquad \dfrac{\vdash A^{\perp}, ?\Delta}{\vdash !A^{\perp}, ?\Delta}\,(!)}{\vdash \Gamma, ?\Delta}\,(cut) \quad \to \quad \dfrac{\vdash \Gamma}{\vdash \Gamma, ?\Delta}\,(?w)} \quad \rightsquigarrow \quad \dfrac{\dfrac{\vdash \Gamma}{\vdash A, \Gamma}\,(w) \qquad \vdash A^{\perp}, \Delta}{\vdash \Gamma, \Delta}\,(cut) \quad \to \quad \dfrac{\vdash \Gamma}{\vdash \Gamma, \Delta}\,(w)$$

$$\cancel{\dfrac{\dfrac{\vdash A, \Gamma}{\vdash A, ?B, \Gamma}\,(?w) \qquad \vdash A^{\perp}, \Delta}{\vdash \Gamma, ?B, \Delta}\,(cut) \quad \to \quad \dfrac{\dfrac{\vdash A, \Gamma \qquad \vdash A^{\perp}, \Delta}{\vdash \Gamma, \Delta}\,(cut)}{\vdash \Gamma, ?B, \Delta}\,(?w)} \quad \rightsquigarrow \quad \dfrac{\dfrac{\vdash A, \Gamma}{\vdash A, B, \Gamma}\,(w) \qquad \vdash A^{\perp}, \Delta}{\vdash \Gamma, B, \Delta}\,(cut) \quad \to \quad \dfrac{\dfrac{\vdash A, \Gamma \qquad \vdash A^{\perp}, \Delta}{\vdash \Gamma, \Delta}\,(cut)}{\vdash \Gamma, B, \Delta}\,(w)$$

Lemma

Cut-elimination in Affine Logic is weakly normalizing.

Proof.

Simple adaptation of such a proof for Linear Logic. □

Back-up: No Uniform Interpolation in Affine Logic

Lemma

In Affine Logic there is no proof of
$X^a, (X \multimap (X \otimes Y))^b, (!(X \multimap (X \otimes Y)))^c, (X \otimes Y)^d, Y^e, (X \multimap Y)^f \vdash$

Proof.

By contradiction, consider a smallest cut-free proof. Its last rule is either a $\otimes$- $\parr$- $?d$- $?c$- or w-rule, with as premise a sequent of the same kind. □

Proposition

(Intuitionistic) Affine Logic does not have uniform interpolation.

Proof.

Same proof as for Linear Logic, except we use the above lemma so as to have an ax-rule on each Y^+ of B_n, hence an alternating path in our GOI projection graph. □