From the concept of computation to the continuum problem: Gödel's dialectic of the formal and the informal.

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Directed by Gabriella Crocco (CGGG) and Lionel Nguyen Van The (I2M)

Stating the problem

"a complete solution of these problems can be obtained only by ***a more profound analysis (than mathematics is accustomed to give) of the meanings of the terms occurring in them (such as "set", "one-to-one correspondence", etc.) and of the axioms underlying their use****."*

[Gödel 1947 (1990)]

Plan of the talk

1. Gödel's philosophy and his conceptual analysis of the concept of computability.
2. From the concept of computability to the concept of set.
3. Applying Gödel's analysis to contemporary set theory.

Kurt Gödel (1906-1978)

"Classes and concepts may, however, also be conceived as real objects, namely classes as "pluralities of things" or as structures consisting of a plurality of things and concepts as the properties and relations of things existing independently of our definitions and constructions." ***[Gödel 1944]***

"Generally the range of applicability of a concept need not form a set. An obvious example is the concept of set, whose range consists of all sets. A familiar and convenient practice is to take the range of any concept as a class. When the range of a concept is a set, the set is its extension. Since, strictly speaking, an extension should be one object but a class which is not a set is not one object, we can generally speak of the extension of a concept only as a façon de parler." Gödel, quoted in ***[Wang 1996]***

"My philosophical point of view:
[...]
2. Human reason can, in principle, be developed more highly (through certain techniques)
3. There are systematic methods for the solution of all problems"

[2016 Crocco, Engelen], p. 36

Let S be a first-order, recursively enumerable and consistent formal system at least as expressive as Peano's arithmetic.

1st incompleteness theorem

There is a sentence A such that $S \nvdash A$ and $S \nvdash \neg A$.

2nd incompleteness theorem

$Con(S)$ is one of these sentences.

"My incompleteness theorem makes it likely that mind is not mechanical, or else mind cannot understand its own mechanism. If my result is taken together with the rationalistic attitude which Hilbert had and which was not refuted by my results, then [we can infer] the sharp result that mind is not mechanical. This is so, because, if the mind were a machine, there would, contrary to this rationalistic attitude, exist number-theoretic questions undecidable for the human mind."

Gödel, quoted in **[Wang 1996]**

1 Gödel: the conceptual analysis of the concept of computability

> *"Indeed, you are of the opinion (just as I am) that* ***an exhaustive definition of the notion 'finitary proof' is impossible****."*

Letter from Gödel to Herbrand, 1931

1 Gödel: the conceptual analysis of the concept of computability

> *"In consequence of later advances, in particular of the fact that due to A. M. Turing's work a precise and unquestionably adequate definition of the general notion of formal system can now be given, a completely general version of Theorems VI and XI is now possible. "*

Gödel, note added in 1963 to his 1931 Incompleteness article

1 Gödel: the conceptual analysis of the concept of computability

"7.3.10 We begin with vague perceptions of a concept, as we see an animal far away or take two stars for one before using the telescope. For example, we had the precise concept of mechanical procedure in mind, but had not perceived it clearly before we knew of Turing's work."

[Wang 1996], p. 235

"[...] with [Turing's definition of computability] ***one has for the first time succeeded in giving an absolute definition of an interesting epistemological notion,*** *i.e., one not depending on the formalism chosen. [...] By a kind of* ***miracle*** *it is not necessary to distinguish orders, and the diagonal procedure does not lead outside the defined notion. This, I think, should encourage one to* ***expect the same thing to be possible also in other cases (such as demonstrability or definability)****. It is true that for these other cases there exist certain negative results, such as the incompleteness of every formalism or the paradox of Richard. But closer examination shows that these results do not make a definition of the absolute notions concerned impossible under all circumstances, but only exclude certain ways of defning them."*

[Gödel 1946 (1990), 150]

1 Gödel: Absoluteness, independence of language

Closure under diagonal process

The class of the formalised objects is closed by application of a diagonal procedure.

Confluence

A notion is confluent when one can prove that different formalisations of it are extensionally equivalent.

Grounding

"*Among the class of conceptually distinct precisifications of the given (intuitive) concept, one of them stands out as being indubitably adequate – as being the right idea.*"
[Kennedy 2020].

1 Gödel: Absoluteness, independence of the domain of application

Acts

When defining the notion, we do not focus on the objects which satisfy the definition, but on the acts which using the notion amounts to.

Stability under generalisation

The scope of the definition corresponds to the extension of the defined concept; it cannot be extended without contradicting our intensional understanding of the defined concept. In particular, any generalisation of the definition, by any conceivable means, which would lead to an extension of the class of defined objects, can be recognised as incorrect in relation to the defined concept.

2 From the concept of computability to the concept of set

"with this concept [Turing's computability] one has for the first time succeeded in giving an absolute definition of an interesting epistemological notion, i.e., one not depending on the formalism chosen. [...] This, I think, should encourage one to expect the same thing to be possible also in other cases (such as demonstrability or definability)."

[Gödel 1946]

Explicit definitions of concepts are in many cases impossible, because for the higher concepts (ex. mathematics):

> *"their meaning may be undefinable (i.e. irreducible to anything more fundamental)". [...]*
> *"the meaning of a concept cannot in general be expressed by a set of formal rules"*

[Gödel 1944] p139

> *"It is true that for these other cases there exist certain negative results, such as the incompleteness of every formalism or the paradox of Richard. But closer examination shows that these results do not make a definition of the absolute notions concerned impossible under all circumstances, but only exclude certain ways of defning them"*

[Gödel 1946]

"On the whole, Gödel seems to favor the fundamental perspective of seeing objective reality, both physical and conceptual as eternal, timeless and fixed. At the same time, he believes that is possible for us at least partially and step by step [...] to approach closer the objective reality itself"

[Wang 1996, p. 322]

" Let us consider, e.g., the concept of demonstrability. It is well known that, in whichever way you make it precise by means of a formalism, the contemplation of this very formalism gives rise to new axioms which are exactly as evident and justified as those with which you started and that this process of extension can be iterated into the transfinite. So ***there cannot exist any formalism which would embrace all these steps; but this does not exclude that all these steps*** *(or at least all of them which give something new for the domain of propositions in which you are interested)* ***could be described and collected together in some non-constructive way****. In set theory, e.g., the successive extensions can most conveniently be represented by stronger and stronger axioms of infinity."*

[Gödel 1946]

The axiom of inaccessible cardinals states that there is a set that is closed for the power set operation.

Inaccessible cardinals

Let κ an infinite, uncountable, regular cardinal. κ is (strongly) inaccessible if for every cardinal α:

$$(\alpha < \kappa) \Rightarrow (|\mathcal{P}(\alpha)| < \kappa)$$

The existence of a large cardinal κ is independent of ZFC and implies that V_κ is a model of ZFC.

"It is certainly impossible to give a combinational and decidable characterization of what an axiom of infinity is; but there might exist, e.g., a characterization of the following sort: An axiom of infinity is a proposition which has a certain (decidable) formal structure and which in addition is true."

[Gödel 1946 (1990), 150]

2 The concept of set: Absolute demonstrability

"Reflection principle : the universe of all sets is structurally undefinable. One possibility of making this statement precise is the following : The universe of sets cannot be uniquely characterized (i.e. distinguished from all its initial segments) by any internal structural property of the $\in$-relation in it, expressible in any logic of finite or transfinite type, including infinitary logics of any cardinal number."

[Wang 1974]

In ZFC, every cardinality can be associated with one of the cardinals $\aleph_\alpha$, where α is an ordinal (so, for example, one can define $\aleph_\omega$). The cardinal $\aleph_\alpha$ is the least cardinal greater than all preceding cardinals.

- $\aleph_0 = \omega$ is the cardinality of countable sets; it is the least infinite cardinal.
- $\aleph_1$ is the least uncountable cardinal; it is the set of all countable ordinals.
- We only know that $|\mathbb{R}| = 2^{\aleph_0} \geq \aleph_1$.
- Cantor's Continuum Hypothesis (CH) states that $2^{\aleph_0} = \aleph_1$.

- If CH is true, then every infinite subset E of $\mathbb{R}$ is either countable or has the cardinality of the continuum.
- If CH is false, then there must exist counterexamples, i.e. subsets E of $\mathbb{R}$ such that $\aleph_0 < |E| < 2^{\aleph_0}$.

- Open subsets of $\mathbb{R}$, as well as closed subsets, satisfy the CH.
- By taking countable unions and complements of open and closed sets, we obtain the Borel sets, which satisfy the CH.
- The images of Borel sets under continuous functions are the analytic sets, which also satisfy the CH.
- The complements of analytic sets are not always analytic and are called co-analytic.
- By taking complements and images under continuous functions of analytic sets, we obtain the projective sets.

2 The concept of set:The Continuum Hypothesis

"Il n'y a là qu'une seule lacune importante : on ne sait pas si tout complémentaire analytique (c'est-à-dira le complémentaire d'un ensemble analytique) non dénombrable a la puissance du continu.
Les efforts que j'ai faits pour résoudre cette question m'ont conduit à ce résultat tout inattendu : ***il existe une famille*** *admettant une application sur le continu d'ensembles effectifs* ***telle qu'on ne sait pas et l'on ne saura jamais si un ensemble quelconque de cette famille (supposé non dénombrable) a la puissance du continu****, s'il est ou non de troisième catégorie, ni même s'il est mesurable."*
[Luzin 1925]

Is CH decidable from ZFC axioms ?

- In 1947, Gödel argued for the undecidability of CH in his paper *What is Cantor's Continuum Problem?*
- Gödel showed in 1940 that if ZFC is coherent then $ZFC + (V = L)$ is coherent. Since $ZFC + (V = L)$ implies CH, that implies that if ZFC is coherent then ZFC can't refute CH.

"It is to be noted, however, that, even if one should succeed in proving its undemonstrability as well, this would (in contradistinction, for example, to the proof for the transcendency of π) by no means settle the question definitively. Only someone who (like the intuitionist) denies that the concepts and axioms of classical set theory have any meaning (or any well-defined meaning) could be satisfied with such a solution, not ***someone who believes them to describe some well-determined reality****. For in this reality Cantor's conjecture must be either true or false, and its undecidability from the axioms as known today can only mean that these axioms do not contain a complete description of this reality;"*

[Gödel 1947]

"there are two quite differently defined classes of objects which both satisfy all axioms of set theory written down so far. One class consists of the sets definable in a certain manner by properties of their elements, the other of the sets in the sense of arbitrary multitudes irrespective of if, or how, they can be defined. [...] This characteristic of sets, however, is neither formulated explicitly nor contained implicitly in the accepted axioms of set theory"

[Gödel 1947]

L : Constructible sets

L is defined by transfinite recursion as follows:

- $L_0 = \varnothing$;
- $L_{\alpha+1} = \texttt{Def}(L_\alpha)$;
- $L_\beta = \bigcup_{\alpha<\beta} L_\alpha$, when β is a limit ordinal.

Here, $\texttt{Def}(E) = \{X \subseteq E \mid z \in X \Leftrightarrow \vDash_{\langle E,\in\rangle} \Phi(z, y_1, ..., y_n)\}$ for Φ first-order formula and $y_1, ..., y_n \in E$.

2 The concept of set: Definability

OD : Ordinal Definable sets

The **OD** sets are the sets such that:

$$\forall x(\Phi(x, \alpha_1, ..., \alpha_n) \Leftrightarrow x = a)$$

where Φ is a first-order formula and $\alpha_1, ..., \alpha_n$ are ordinals.

HOD : Hereditarily Ordinal Definable sets

HOD = transitive closure of **OD**

"I would think that "definability in terms of ordinals", even if it is not an adequate formulation for "comprehensibility by our mind", is at least an adequate formulation in an absolute sense for a closely related property of sets, namely, the property of "being formed according to a law" as opposed to "being formed by a random choice of the elements".

[Gödel 1946 (1990)], p150

2 The concept of set: Definability

> *"It is not in the ideas (of set and concept) themselves that every set is the extension of a concept. Sets might exist which correspond to no concepts. The "proposition for every set, there is a [defining] concept" requires a proof. But I conjecture that it is true. If so, everything (in logic and mathematics) is a concept : a set, if extensionnal; and a concept (only) otherwise."*

[Wang 1996], 8.6.4 p274

Is CH decidable from ZFC axioms ?

- Gödel showed in 1940 that if ZFC is coherent then $ZFC + (V = L)$ is coherent. Since $ZFC + (V = L)$ implies CH, that implies that if ZFC is coherent then ZFC can't refute CH.
- Cohen showed in 1964 that if ZFC is coherent then $ZFC + \neg CH$ is coherent
- **Gödel's Program**: the search for new axioms extending ZFC and deciding statements independent of ZFC, including (but not limited to) the Continuum Hypothesis.

Substantial progress has been made along the lines suggested by Gödel:

- Forcing
- Inner Model Program
- Developments in Large Cardinal Theory

3 Applying Gödel's analysis to contemporary set theory: Forcing

Forcing allows us to easily change the cardinality of the continuum:

Easton's Theorem

Let M be a transitive model of ZFC+GCH and let f be a function defined on the regular cardinals and taking cardinal values. Assume that for every regular cardinal κ and λ:

1. $\kappa < f(\kappa)$
2. *If* $\kappa \leq \lambda$ *then* $f(\kappa) \leq f(\lambda)$
3. $\kappa < cof(f(\kappa))$

Then there exists a generic extension $M[G]$ of M such that $M[G]$ satisfies ZFC and $M[G] \vDash [2^{\kappa} = f(\kappa)]$

3 Applying Gödel's analysis to contemporary set theory: Forcing

Forcing allows us to easily change the cardinality of the continuum, without perturbing the large cardinal hierarchy.

Lévy Collapse Theorem

Let κ be a regular cardinal and $\lambda > \kappa$ an inaccessible cardinal[a]*. There exists a forcing notion $\mathbb{P}$ such that:*

1. *All ordinals α such that $\kappa \leq \alpha < \lambda$ have cardinality κ in $M[G]$ and*
2. *All ordinals β of M such that $\beta \leq \kappa$ or $\beta \geq \lambda$ remain cardinals in $M[G]$.*

[a] A cardinal κ is inaccessible if it is regular, uncountable, and for all $\lambda < \kappa$ we have $2^\lambda < \kappa$.

3 Applying Gödel's analysis to contemporary set theory: Hamkins Multiverse

"I shall argue for a contrary position, the multiverse view, which holds that there are diverse distinct concepts of set, each instantiated in a corresponding set-theoretic universe, which exhibit diverse set-theoretic truths. Each such universe exists independently in the same Platonic sense that proponents of the universe view regard their universe to exist.
[...]
Often, the clearest way to refer to a set concept is to describe the universe of sets in which it is instantiated, and in this article I shall simply identify a set concept with the model of set theory to which it gives rise."

[Hamkins 2012]

3 Applying Gödel's analysis to contemporary set theory: Inner Model Program

- **Theorem (Scott 1961) :** The existence of measurable cardinals contradicts the axiom of constructibility $V = L$.
- The **Inner Model Program** aims to construct models resembling L while accommodating large cardinal axioms.

3 Applying Gödel's analysis to contemporary set theory: Inner Model Program

> *" the results presented in this paper suggest the intriguing possibility that one can reduce the inner model problem for all known large cardinal axioms to the inner model problem for exactly one supercompact cardinal."*

p141 **[Woodin 2010]**

3 Applying Gödel's analysis to contemporary set theory: Large Cardinals

- **Definition**: A measurable cardinal is an uncountable cardinal κ such that there exists a non-trivial, κ-additive, $\{0,1\}$-valued measure μ defined on all subsets of κ. The definition of measurable cardinals by Ulam in 1930 arose from the study of the possibility of extending the Lebesgue measure to any set of reals.
- **Theorem** (1930 Tarski-Ulam): If κ is measurable, then κ is inaccessible.
- **Theorem** (1964 Hanf): The smallest inaccessible cardinal is not measurable, i.e., the property of measurability is stronger than inaccessibility.
- By reformulating the notion of measurability in terms of elementary embedding, Scott proved the following result: **Theorem** (Scott 1961): The existence of a measurable cardinal contradicts the axiom of constructibility $V = L$.

3 Applying Gödel's analysis to contemporary set theory: Large Cardinals

The determination property stems from game theory; it asserts the existence of a winning strategy for certain games defined on sets of real numbers.

- **Definition**: AD is the axiom "Every set of reals is determined."
- **Theorem**: AD implies that every set of reals is Lebesgue measurable, has the Baire property, and has the perfect set property.
- **Theorem** (Steinhaus 1965): AD contradicts the Axiom of Choice.
- Therefore, restrictions of AD are formulated (in particular, $AD^{L(\mathbb{R})}$ and PD).
- **Theorem** (Woodin 1985): The existence of a measurable cardinal larger than all Woodin cardinals implies $AD^{L(\mathbb{R})}$.

3 Applying Gödel's analysis to contemporary set theory: Justification

Distinction between **intrinsic** and **extrinsic** justifications:

> *" even disregarding the intrinsic necessity of some new axiom, and even in case it has no intrinsic necessity at all, a probable decision about its truth is possible also in another way, namely, inductively by studying its 'success'. "*

[Gödel 1947]

3 Applying Gödel's analysis to contemporary set theory: Justification

Maddy views intrinsic and extrinsic justifications as distinct and in competition:

> *" intrinsic considerations are valuable, but only insofar as they correlate with these extrinsic payoffs. This suggests that the importance of intrinsic considerations is merely instrumental, that the fundamental justificatory force is all extrinsic. This casts serious doubt on the common opinion that intrinsic justifications are the grand aristocracy and extrinsic justifications the poor cousins. The truth may well be the reverse "*

[Maddy 2011]

3 Applying Gödel's analysis to contemporary set theory: Justification

"Whilst there is a sharp distinction between a principle itself and its consequences, this is not so for the justificatory force attaching to intrinsic and extrinsic justifications. [...] The key issue is that the justificatory force of intrinsic and extrinsic justifications seems, to our minds, not to be sufficiently well-separated. Thus an account of justification that sees one kind as privileged fails to be transparent with respect to the grounds of asserting the 'axiom'."

[Barton, Ternullo, Venturi 2020]

Thank You !

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