Towards a unified perspective in circuit complexity via discrete ODEs

(joint works with A. Durand, J. Kontinen, R. Li)

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IRN $\langle$L|I$\rangle$

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ROADMAP

- INTRODUCTION
- FROM BOUNDED RECURSION TO DISCRETE ODES
- SMALL CIRCUIT CLASSES (AND CORRESPONDING ALGEBRAS)
- A NEW FRAMEWORK IN CIRCUIT COMPLEXITY
- FUTURE PERSPECTIVES

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INTRODUCTION

(IMPLICIT) COMPUTATIONAL COMPLEXITY

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J. Hartmanis & R. Stearns, *On the Computational Complexity of Algorithms*, Trans. Amer. Math. Soc., 117, pp. 285-306, 1965.

IMPLICIT COMPUTATIONAL COMPLEXITY

- recursion theory

restricting the primitive recursion schema

IMPLICIT COMPUTATIONAL COMPLEXITY

- recursion theory
- descriptive complexity

considering expressibility in logical systems

IMPLICIT COMPUTATIONAL COMPLEXITY

- recursion theory
- descriptive complexity
- proof theory

relying on the correspondence between proofs and programs

IMPLICIT COMPUTATIONAL COMPLEXITY

- **recursion theory**
- descriptive complexity
- proof theory

A NEW FRAMEWORK IN COMPLEXITY

A NEW FRAMEWORK IN COMPLEXITY

THE CONTEXT : IMPLICIT COMPLEXITY

$$\mathbf{FP} = \mathcal{A}_{Cob}$$

functions computable in polynomial time

based on bounded recursion on notation [Cobham64]

A NEW FRAMEWORK IN COMPLEXITY

THE CONTEXT : IMPLICIT COMPLEXITY

$$\textbf{FP} = \mathbb{LDL}$$

functions computable in polynomial time

based on the "linear-length ODE" schema [BournezDurand19]

A NEW FRAMEWORK IN COMPLEXITY

THE GOAL

(Implicit) characterizations of small circuit classes based on discrete ODE-schemas.

$\mathbf{FAC}^k$, $\mathbf{FTC}^k$, $\mathbf{FNC}^k$

A NEW FRAMEWORK TO COMPLEXITY

$$\mathbf{FP} = \mathbb{LDL} \quad \rightsquigarrow \quad \mathbf{FNC}^k, \mathbf{FAC}^k, \mathbf{FTC}^k = ?$$

FROM BOUNDED RECURSION TO DISCRETE ODES

Starting point

$$\mathbf{FP} = \mathbb{LDL} \quad \rightsquigarrow \quad \mathbf{FNC}^k, \mathbf{FAC}^k, \mathbf{FTC}^k = ?$$

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STARTING POINT

O. Bournez and A. Durand, *Recursion Schemes, Discrete Differential Equations and Characterization of Polynomial Time Computations*, MFCS, 23, pp. 1-14, 2019.
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Discrete ordinary differential equations...

- natural tool for algorithm design
- (implicit) characterization for complexity classes

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Discrete ordinary differential equations...

- natural tool for algorithm design
- (implicit) characterization for complexity classes (in particular for **FP**).

CAPTURING CLASSES VIA RECURSION SCHEMAS

CAPTURING CLASSES VIA RECURSION SCHEMAS

PRIMITIVE RECURSION (PR)

$$f(0, \mathbf{y}) = g(\mathbf{y})$$
$$f(x+1, \mathbf{y}) = h(x, \mathbf{y}, f(x, \mathbf{y}))$$

$$\mathbf{PR} = [0, \pi_i^p, s; \circ, \mathrm{PR}]$$

Capturing complexity via limited recursion schemas

Primitive recursion (PR)

$$f(0, \mathbf{y}) = g(\mathbf{y})$$
$$f(x+1, \mathbf{y}) = h(x, \mathbf{y}, f(x, \mathbf{y}))$$

restrictions on recursion to capture complexity classes

$$\mathbf{PR} = [0, \pi_i^p, s; \circ, \mathrm{PR}]$$

Capturing complexity via limited recursion schemas

Bounded recursion on notation (BRN)

$i \in \{0, 1\}$

$$
\begin{aligned}
f(0, \mathbf{y}) &= g(\mathbf{y}) \\
f(s_i(x), \mathbf{y}) &= h_i(x, \mathbf{y}, f(x, \mathbf{y})) \\
f(x, \mathbf{y}) &\leq k(x, \mathbf{y}) \quad \text{for all } x, \mathbf{y}
\end{aligned}
$$

- explicit bound
- binary successor function

Capturing complexity via limited recursion schemas

Bounded recursion on notation (BRN)

$i \in \{0, 1\}$

does not grow too fast

$$f(0, \mathbf{y}) = g(\mathbf{y})$$
$$f(s_i(x), \mathbf{y}) = h_i(x, \mathbf{y}, f(x, \mathbf{y}))$$
$$f(x, \mathbf{y}) \leq k(x, \mathbf{y}) \quad \text{for all } x, \mathbf{y}$$

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$s_i(x) = 2x + i$

- explicit bound
- binary successor function

control the number of recursion steps

Capturing complexity via limited recursion schemas

Bounded recursion on notation (BRN)

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\end{aligned}
$$

- explicit bound
- binary successor function

$x\#y = 2^{\ell(x)\times\ell(y)}$

$$\mathbf{FP} = [0, \pi_i^p, s_0, s_1, \#; \circ, \mathrm{BRN}]$$

FROM BOUNDED RECURSION TO ODE

Discrete ODE-based characterization of **FP** :

- no explicit bound in the recursion
- no specific role or type for variables

From bounded recursion to ODE

$\Delta f(x) = f(x+1) - f(x)$

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as instead based on

- derivation along function
- linearity

FROM BOUNDED RECURSION TO ODE

Discrete ODE-based characterization of **FP** :

- no explicit bound in the recursion
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to control the number of computation steps

as instead based on

- derivation along function : $\mathcal{L}$-ODE
- linearity

FROM BOUNDED RECURSION TO ODE

Discrete ODE-based characterization of **FP** :

- no explicit bound in the recursion
- no specific role or type for variables

as instead based on

- derivation along function : $\mathcal{L}$-ODE

  to control the object size
- linearity : (essentially) linear ODEs

CAPTURING FP : LINEAR $\mathcal{L}$-ODE

DEFINITION (LINEAR $\mathcal{L}$-ODE). Function f is linear $\mathcal{L}$-ODE definable from u, g and h if it is the solution of IVP :

$$f(0, \mathbf{y}) = g(\mathbf{y})$$
$$\frac{\partial f(x, \mathbf{y})}{\partial \mathcal{L}} = u\big(x, \mathbf{y}, f(x, \mathbf{y}), h(x, \mathbf{y})\big),$$

where u is ESSENTIALLY LINEAR in $f(x, \mathbf{y})$.

CAPTURING FP : LINEAR $\mathcal{L}$-ODE

DEFINITION (LINEAR $\mathcal{L}$-ODE). Function f is linear $\mathcal{L}$-ODE definable from u, g and h if it is the solution of IVP :

The discrete derivative of $f(x)$ is $f(x+1) - f(x)$

$$f(0, \mathbf{y}) = g(\mathbf{y})$$

$$\frac{\partial f(x, \mathbf{y})}{\partial \mathcal{L}} = u\big(x, \mathbf{y}, f(x, \mathbf{y}), h(x, \mathbf{y})\big),$$

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where u is ES

$$f(x+1,\mathbf{y}) = f(x,\mathbf{y}) + (\mathcal{L}(x+1,\mathbf{y}) - \mathcal{L}(x,\mathbf{y})) \times u(x,\mathbf{y}, f(x,\mathbf{y}), h(x,\mathbf{y}))$$

Capturing FP : linear $\mathcal{L}$-ODE

Definition (Linear $\mathcal{L}$-ODE). Function f is linear $\mathcal{L}$-ODE definable from u, g and h if it is the solution of IVP :

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where u is ESS

The value of $f(x, \mathbf{y})$ changes only when the value of $\mathcal{L}(x, \mathbf{y})$ does.

CAPTURING FP : LINEAR $\mathcal{L}$-ODE

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where u is ESSENTIALLY LINEAR in $f(x, \mathbf{y})$.

CAPTURING FP : LINEAR $\mathcal{L}$-ODE

DEFIN[...] d h if it
is the

A sg-polynomial expression is an expression over the signature $\{+, -, \times, \mathrm{sg}\}$.
A sg-polynomial expression P is ESSENTIALLY LINEAR IN x if $\mathbf{deg}(x, P) = 1$, i.e. there exist sg-expressions Q_1, Q_2 such that $P = Q_1 \times x + Q_2$ and $\mathbf{deg}(x, Q_1) = \mathbf{deg}(x, Q_2) = 0$.
The degree $\mathbf{deg}(x, P)$ of a term $x \in X$ in P :

- $\mathbf{deg}(x, x) = 1$ and, for $x' \in X \cup \mathbb{Z}$ s.t. $x' \neq x$, $\mathbf{deg}(x, x') = 0$
- $\mathbf{deg}(x, P + Q) = \mathbf{deg}(x, P - Q) = \max\{\mathbf{deg}(x, P), \mathbf{deg}(x, Q)\}$
- $\mathbf{deg}(x, P \times Q) = \mathbf{deg}(x, P) + \mathbf{deg}(x, Q)$
- $\mathbf{deg}(x, \mathrm{sg}(P)) = 0$.

$\partial\mathcal{L}$

where u is ESSENTIALLY LINEAR in $f(x, \mathbf{y})$.

CAPTURING FP : LINEAR $\mathcal{L}$-ODE

A sg-polynomial expression is an expression over the signature $\{+, -, \times, \mathrm{sg}\}$.
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DEFIN... d h if it is the ...

$\partial\mathcal{L}$

where u is ESSENTIALLY LINEAR in $f(x, \mathbf{y})$.

Capturing FP : linear $\mathcal{L}$-ODE

Definition (Linear $\mathcal{L}$-ODE). Function f is linear $\mathcal{L}$-ODE definable from u, g and h if it is the solution of IVP :

$$\begin{aligned} f(0,\mathbf{y}) &= g(\mathbf{y}) \\ \frac{\partial f(x,\mathbf{y})}{\partial \mathcal{L}} &= A(x,\mathbf{y},f,h) \times f(x,\mathbf{y}) + B(x,\mathbf{y},f,h) \end{aligned}$$

where $\mathbf{u}$ occurs in A and B under the scope of sg.

CAPTURING FP : LINEAR ℓ-ODE

DEFINITION (LINEAR LENGTH-ODE, ℓ-ODE). Function f is linear length-ODE definable from u, g and h, if it is the solution of IVP :

$$
\begin{aligned}
f(0, \mathbf{y}) &= g(\mathbf{y}) \\
\frac{\partial f(x, \mathbf{y})}{\partial \ell} &= u\big(x, \mathbf{y}, f(x, \mathbf{y}), h(x, \mathbf{y})\big),
\end{aligned}
$$

where u is ESSE

$\ell(x)$ returns the length of x written in binary

CAPTURING FP : LINEAR ℓ-ODE

DEFINITION (LINEAR LENGTH-ODE, ℓ-ODE). Function f is linear length-ODE definable from u, g and h, if it is the solution of IVP :

$$f(0, \mathbf{y}) = g(\mathbf{y})$$

$$\frac{\partial f(x, \mathbf{y})}{\ldots} = u(x, \mathbf{y}, f(x, \mathbf{y}), h(x, \mathbf{y}))$$

where u is ESSENTIAL…

The solution of this system is

$$f(x, y) = \sum_{u=-1}^{\ell(x)-1} \left(\prod_{t=u+1}^{\ell(x)-1} (1 + A(\alpha(t), \mathbf{y}) \right) \times B(\alpha(u), \mathbf{y})$$

where $\alpha(z) = 2^z - 1$, $\prod_x^{x-1} \kappa(x) = 1$ and $B(\alpha(-1), \cdot) = f(0, \mathbf{y})$.

CAPTURING FP : $\mathbb{LDL}$

LINEAR ℓ-ODE

$$\frac{\partial f(x,\mathbf{y})}{\partial \ell} = u(f,h,x,\mathbf{y})$$

- derivation along to the length function ℓ
- linearity

$$\mathbf{FP} = [0,1,\ell,\mathrm{sg},+,-,\times,\pi^p_i;\circ,\ell\text{-ODE}]$$

CAPTURING FP : $\mathbb{LDL}$

LINEAR ℓ-ODE

$$\frac{\partial f(x, \mathbf{y})}{\partial \ell} = u(f, h, x, \mathbf{y})$$

- derivation along to the length function ℓ
- linearity

$\mathrm{sg}(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$

$$\mathbf{FP} = [0, 1, \ell, \mathrm{sg}, +, -, \times, \pi_i^p; \circ, \ell\text{-ODE}]$$

$\ell(0) = 0$ and $\ell(x) = \lceil \log_2(x+1) \rceil$

composition

CAPTURING FP : $\mathbb{LDL}$

LINEAR ℓ-ODE

$$\frac{\partial f(x, \mathbf{y})}{\partial \ell} = u(f, h, x, \mathbf{y})$$

to control the number of computation steps

- derivation along to the length function ℓ
- linearity

to control the object size

$$\mathbf{FP} \supseteq [\mathbf{0}, \mathbf{1}, \ell, \mathrm{sg}, +, -, \times, \pi_i^p; \circ, \ell\text{-ODE}]$$

CAPTURING FP : $\mathbb{LDL}$

LINEAR ℓ-ODE

$$\frac{\partial f(x, \mathbf{y})}{\partial \ell} = u(f, h, x, \mathbf{y})$$

- derivation along to the length function ℓ
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$$\mathbf{FP} \subseteq [0, 1, \ell, \mathrm{sg}, +, -, \times, \pi_i^p; \circ, \ell\text{-ODE}]$$

"encoding" computation of a poly-time register machine via ℓ-ODE

ON SMALL CIRCUIT CLASSES (AND CORRESPONDING ALGEBRAS)

Capturing small circuit classes

$$\mathbf{FP} = \mathbb{LDL} \quad \rightsquigarrow \quad \mathbf{FNC}^k, \mathbf{FAC}^k, \mathbf{FTC}^k = ?$$

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ON PARALLEL COMPUTATION : INTUITION

In parallel computation multiple operations can be executed simultaneously

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A parallel algorithm is efficient when :

ON PARALLEL COMPUTATION : INTUITION

In parallel computation multiple operations can be executed simultaneously

A parallel algorithm is efficient when :

i.e. polynomial in the length of the input

- the number of processors is not too high

On parallel computation : intuition

In parallel computation multiple operations can be executed simultaneously

A parallel algorithm is efficient when :

- the number of processors is not too high
- speed-up with respect to its (efficient) sequential counterpart.

SMALL CIRCUIT CLASSES : $\mathbf{FAC}^i$

DEFINITION (CLASS $\mathbf{FAC}^i$). $\mathbf{AC}^i$ is the class of languages recognized by a **Dlogtime**-uniform family of unbounded Boolean circuit of polynomial size and $O((\log n)^i)$ depth. We denote by $\mathbf{FAC}^i$ the corresponding function class.

circuits over (standard) unbounded fan-in basis $\{\neg, (\wedge^n)_{n\in\mathbb{N}}, (\vee^n)_{n\in\mathbb{N}}\}$

SMALL CIRCUIT CLASSES : $\mathbf{FAC}^0$

DEFINITION (CLASS $\mathbf{FAC}^0$). $\mathbf{AC}^0$ is the class of languages recognized by a **Dlogtime**-uniform family of unbounded Boolean circuit of polynomial size and constant depth. We denote by $\mathbf{FAC}^0$ the corresponding function class.

SMALL CIRCUIT CLASSES : $\mathbf{FNC}^i$

DEFINITION (CLASS $\mathbf{FNC}^i$). $\mathbf{NC}^i$ is the class of languages recognized by a **Dlogtime**-uniform family of bounded Boolean circuit of polynomial size and $O((\log n)^i)$ depth. We denote by $\mathbf{FNC}^i$ the corresponding function class.

circuits over (standard) bounded fan-in basis $\{\neg, \wedge, \vee\}$

SMALL CIRCUIT CLASSES

DEFINITION (CLASSES $\mathbf{FAC}^i$ AND $\mathbf{FNC}^i$). $\mathbf{AC}^i$ (resp., $\mathbf{NC}^i$) is the class of languages recognized by a **Dlogtime**-uniform family of unbounded (resp., bounded) Boolean circuit of polynomial size and $O((\log n)^i)$ depth. We denote by $\mathbf{FAC}^i$ the corresponding function class.

DEFINITION (CLASSES $\mathbf{FACC}[\mathbf{p}]$ AND $\mathbf{FTC}^0$) For $p \in \mathbb{N}$, $\mathbf{ACC}[\mathbf{p}]$ (resp., $\mathbf{TC}^0$) is the class of languages recognized by a uniform unbounded Boolean circuit family *including* MOD *p gates* (resp., MAJ *gates*) of polynomial size and constant depth. We denote by $\mathbf{FACC}[\mathbf{p}]$ (resp., $\mathbf{FTC}^0$) the corresponding function class.

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MOD p outputs the sum of its inputs modulo p

SMALL CIRCUIT CLASSES

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MAJ outputs 1 when the majority of its inputs are 1's

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$\mathbf{FAC}^0 \subsetneq \mathbf{FACC[2]}$

Small circuit classes

DEFINITION (CLASSES $\mathbf{FAC}^i$ AND $\mathbf{FNC}^i$). $\mathbf{AC}^i$ (resp., $\mathbf{NC}^i$) is the class of languages recognized by a **Dlogtime**-uniform family of unbounded (resp., bounded) Boolean circuit of polynomial size and $O((\log n)^i)$ depth. We denote by $\mathbf{FAC}^i$ the corresponding function class.

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ADD, LOGITADD

ITADD, BCOUNT, MULT, SORT

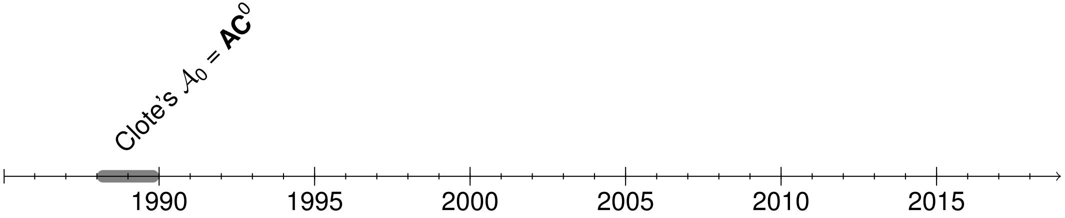
$$\mathbf{FAC}^0 \subsetneq \mathbf{FACC}[2] \subsetneq \mathbf{FTC}^0 \subseteq \mathbf{FNC}^1 \subseteq \mathbf{FAC}^1 \subseteq \mathbf{FNC}^2 \dots$$

PARITY

Capturing small circuit classes

$$\mathbf{FP} = \mathcal{A}_{Cob} \quad \rightsquigarrow \quad \mathbf{FAC}^0, \mathbf{FAC}^k, \mathbf{FNC} = \mathcal{A}_0, \mathcal{A}_k, \mathcal{A}$$

State of the Art



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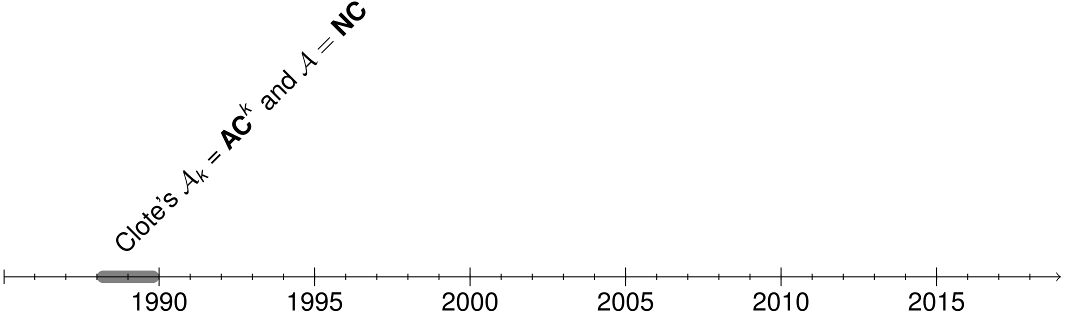

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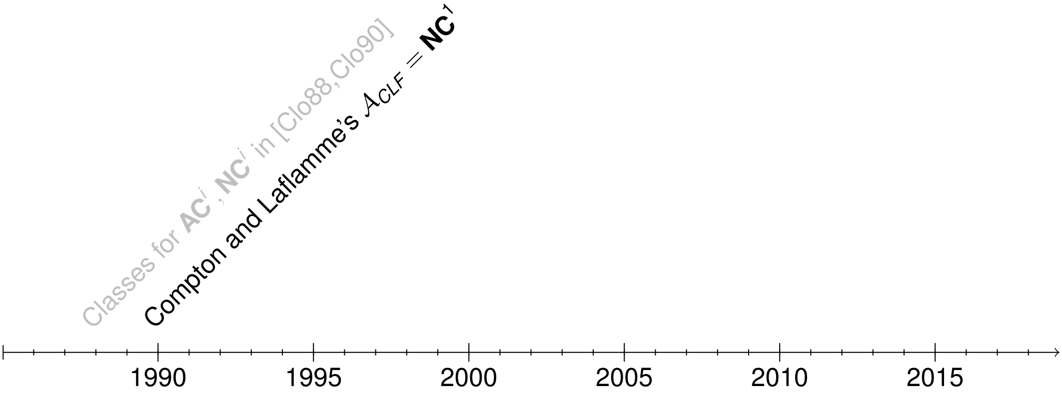

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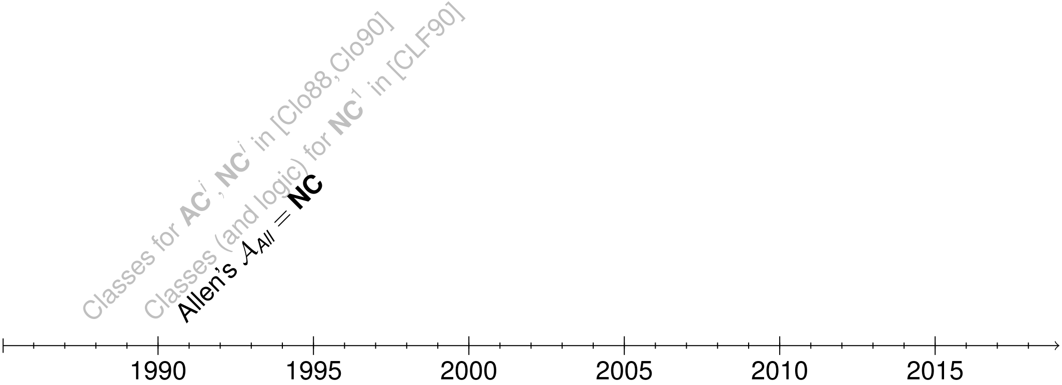

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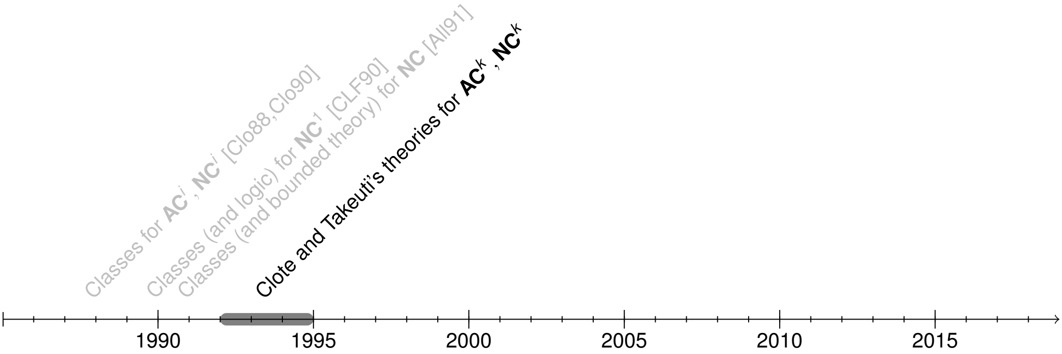
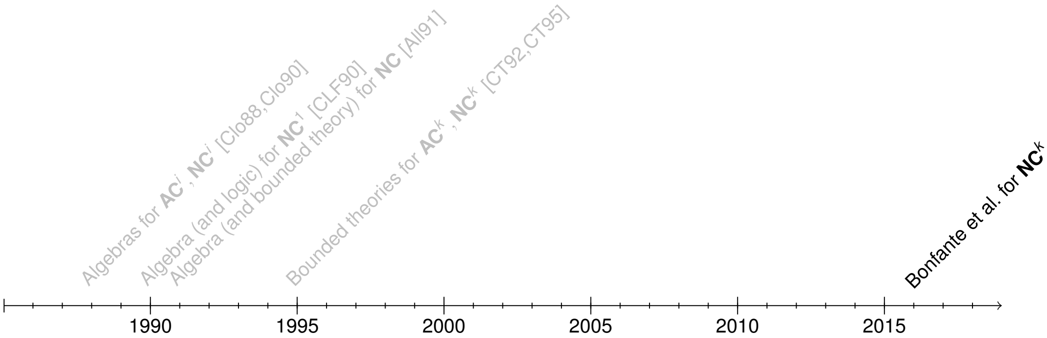


State of the Art



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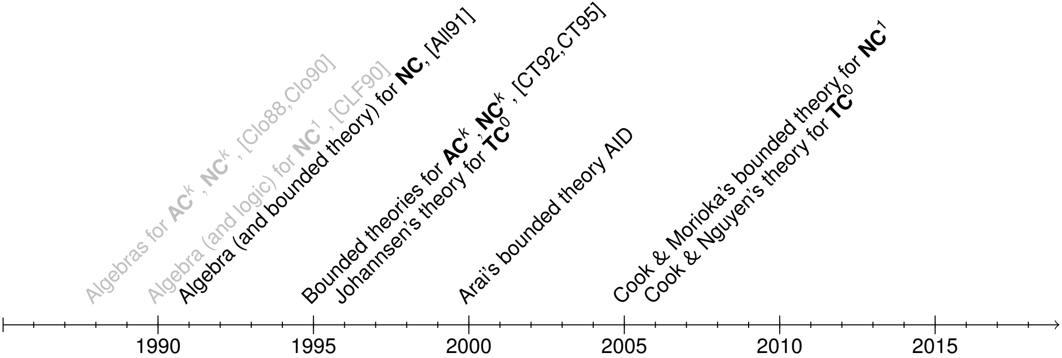

CAPTURING SMALL CIRCUITS VIA LIMITED RECURSION SCHEMAS

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CONCATENATION RECURSION ON NOTATION (CRN)

$i \in \{0,1\}$

$$f(0,\mathbf{y}) = g(\mathbf{y})$$
$$f(s_i(x),\mathbf{y}) = s_{h_i}(f(x,\mathbf{y}))$$

For $h_i(x,\mathbf{y}) \leq 1$

Capturing small circuits via limited recursion schemas

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For $h_i(x, \mathbf{y}) \leq 1$

Clote 1990

$$\mathbf{FAC}^0 = [0, \pi_i^p, s_0, s_1, \mathrm{BIT}, \ell, \#; \circ, \mathrm{CRN}]$$

$s_i(x) = 2x + i$

bit(x,i) yields the i^{th} bit in the binary representation of x

A NEW FRAMEWORK IN CIRCUIT COMPLEXITY

Capturing circuit classes via ODEs

$$\mathbf{FP} = \mathbb{LDL} \quad \rightsquigarrow \quad \mathbf{FNC}^k, \mathbf{FAC}^k, \mathbf{FTC}^k = ?$$

WHY CAPTURING CIRCUIT CLASSES VIA ODEs ?

This framework offers :

- FLEXIBILITY : multiple features we can modify to capture different resources in circuit computation (e.g., constraints on linearity, the function we're deriving along, etc.).
- UNIFORMITY : minimal modification in the basic set of functions and on constraints over discrete ODEs (e.g., over linearity) to jump from one class to the other.

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Reveal a connection between (constraints on) discrete ODEs computation and circuit resources.

Due to this perspective, we obtain :

- Simplified characterizations for multiple classes
- The very first characterizations for some classes (e.g., **FACC[p]** for $p \notin \{2, 6\}$)
- A uniform picture (possibly scaling to the continuous setting)
- Possible connections with induction schema and bounded arithmetic

CASE STUDY : CAPTURING $\mathbf{FAC}^0$

CASE STUDY : CAPTURING FAC^0

THE KEY : A NEW SCHEMA

$$\frac{\partial f(x, \mathbf{y})}{\partial \ell} = f(x, \mathbf{y}) + k(x, \mathbf{y})$$

k takes values in $\{0, 1\}$

to control the number of computation steps

- derivation along the length function ℓ

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- restricted form of linearity

to control the object size

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Example (Function $2^{\ell(x)}$). The function $x \mapsto 2^{\ell(x)}$ can be rewritten as the solution of the IVP below :

$$f(0) = 1$$
$$\frac{\partial f(x)}{\partial \ell} = f(x)$$

CASE STUDY : CAPTURING FAC^0

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The solution of such system is of the form $f(x) = \prod_{u=0}^{\ell(x)} 2 = 2^{\ell(x)}$

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EXAMPLE (FUNCTION $2^{\ell(x)}$). The function $x \mapsto 2^{\ell(x)}$ can be r[…]s the solution of the IVP below :

$g(\mathbf{y}) = 1$

$$f(0) = 1$$

$$\frac{\partial f(x)}{\partial \ell} = f(x)$$

$k(x,\mathbf{y}) = 0$

CASE STUDY : CAPTURING FAC^0

THE KEY : A NEW SCHEMA (ℓ-ODE$_1$)

$$\frac{\partial f(x, \mathbf{y})}{\partial \ell} = f(x, \mathbf{y}) + k(x, \mathbf{y})$$

- derivation along to the length function ℓ
- restricted form of linearity

Intuitively, at each step, we perform a left shift of one unit, possibly adding a bit.

REMARK (SHARPLY BOUNDED QUANTIFICATION). Sharply bounded quantification is captured using the function sg and ℓ-ODE$_1$.

CASE STUDY : CAPTURING $\mathbf{FAC}^0$

THE KEY : A NEW SCHEMA (ℓ-ODE$_1$)

$$\frac{\partial f(x,\mathbf{y})}{\partial \ell} = f(x,\mathbf{y}) + k(x,\mathbf{y})$$

EXAMPLE (SHARPLY BOUNDED EXISTENTIAL QUANTIFICATION). Let $R \subseteq \mathbb{N}^{p+1}$ and h_R be its characteristic function. Then, for all x and $\mathbf{y}$,

$$(\exists z \leq \ell(x))R(z,\mathbf{y}) = \mathrm{sg}(f(x,\mathbf{y}))$$

where f is such that

$$\begin{aligned} f(0,\mathbf{y}) &= h_R(0,\mathbf{y}) \\ \frac{f(x,\mathbf{y})}{\partial \ell} &= f(x,\mathbf{y}) + h_R(\ell(x+1),\mathbf{y}). \end{aligned}$$

CASE STUDY : CAPTURING $\mathbf{FAC}^0$

THE KEY : A NEW SCHEMA (ℓ-ODE$_1$)

$$\frac{\partial f(x,\mathbf{y})}{\partial \ell} = f(x,\mathbf{y}) + k(x,\mathbf{y})$$

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Intuitively, $f(x,\mathbf{y}) \neq 0$ when, for some $z \leq x$, $R(z,\mathbf{y})$ is satisfied, i.e. $h_R(z,\mathbf{y}) = 1$.

$$f(0,\mathbf{y}) = h_R(0,\mathbf{y})$$

$$\frac{f(x,\mathbf{y})}{\partial \ell} = f(x,\mathbf{y}) + h_R(\ell(x+1),\mathbf{y}).$$

CASE STUDY : CAPTURING $\mathbf{FAC}^0$

THE KEY : A NEW SCHEMA (ℓ-ODE$_1$)

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instance of ℓ-ODE$_1$

$$f(0, \mathbf{y}) = h_R(0, \mathbf{y})$$
$$\frac{f(x, \mathbf{y})}{\partial \ell} = f(x, \mathbf{y}) + h_R(\ell(x+1), \mathbf{y}).$$

CASE STUDY : CAPTURING FAC^0

THE KEY IDEA : NEW SCHEMAS

ℓ-ODE$_1$

$$\frac{\partial f(x,\mathbf{y})}{\partial \ell} = f(x,\mathbf{y}) + k(x,\mathbf{y})$$

ℓ-ODE$_3$

$$\frac{\partial f(x,\mathbf{y})}{\partial \ell} = -\left\lceil \frac{f(x,\mathbf{y})}{2} \right\rceil$$

$\lceil \frac{z}{2} \rceil$ is a shorthand for $z - (z \div 2)$

CASE STUDY : CAPTURING $\mathbf{FAC}^0$

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THE FIRST RESULT [MFCS24]

$$\mathbf{FP} = \mathbb{LDL} \quad \rightsquigarrow \quad \mathbf{FAC}^0 = \mathbb{ACDL}$$

$[0, 1, \ell, \mathrm{sg}, +, -, \div 2, \#, \pi^p_i; \circ, \ell\text{-ODE}_1, \ell\text{-ODE}_3]$

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THE FIRST RESULT [MFCS24]

$$\mathbf{FP} = \mathbb{LDL} \quad \rightsquigarrow \quad \mathbf{FAC}^0 = \mathbb{ACDL}$$

($\supseteq$) Basic functions are computable in $\mathbf{FAC}^0$ and closure under composition and ℓ-ODE$_1$

CASE STUDY : CAPTURING $\mathbf{FAC}^0$

THE KEY IDEA : NEW SCHEMAS

ℓ-ODE$_1$

$$\frac{\partial f(x,\mathbf{y})}{\partial \ell} = f(x,\mathbf{y}) + k(x,\mathbf{y})$$

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THE FIRST RESULT [MFCS24]

$$\mathbf{FP} = \mathbb{LDL} \quad \rightsquigarrow \quad \mathbf{FAC}^0 = \mathbb{ACDL}$$

($\subseteq$) Arithmetization (mimicking functions and schemas of Clote's $\mathcal{A}_0$)

CASE STUDY : CAPTURING FAC^0

THE KEY IDEA : NEW SCHEMAS

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THE FIRST RESULT [MFCS24]

$$\mathbf{FP} = \mathbb{LDL} \quad \rightsquigarrow \quad \mathbf{FAC}^0 = \mathcal{A}_0 \subseteq \mathbb{ACDL}$$

$[0, s_0, s_1, \mathrm{BIT}, \ell, \#; \circ, \mathrm{CRN}]$

CASE STUDY : CAPTURING $\mathbf{FAC}^0$

THE KEY IDEA : NEW SCHEMAS

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CAPTURING SMALL CIRCUIT CLASSES VIA ODES

THE KEY : MANY NEW SCHEMAS

$$\frac{\partial f(x, \mathbf{y})}{\partial \ell} = A(x, \mathbf{y}, f, h) \times f(x, \mathbf{y}) + B(x, \mathbf{y}, f, h)$$

- derivation along the length function ℓ
- various constraints on linearity

THE RESULTS [MFCS24, MFCS25, ICALP26]

Uniform ODE characterizations of classes between $\mathbf{FAC}^0$ and $\mathbf{FNC}^1$ (and with counting gates).

Capturing small circuit classes via ODEs

The key : many new schemas

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The results [MFCS24, MFCS25, ICALP26]

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Capturing small circuit classes via ODEs

The key : many new schemas

$$\frac{\partial f(x,\mathbf{y})}{\partial \ell} = A(x,\mathbf{y},f,h)\times f(x,\mathbf{y}) + B(x,\mathbf{y},f,h)$$

- derivation along the length function ℓ
- various constraints on linearity

Some of these classes were never characterized by recursion schemas before; namely, all classes counting modulo n (**FACC**[**n**]) except for **FACC**[**2**] and **FACC**[**6**].

The results [MFCS24, MFC

Uniform ODE characterizations of classes between **FAC**^{0} and **FNC**^{1} (and with counting gates).

THE GLOBAL PICTURE (SO FAR) : FP

BOURNEZ & DURAND 2019

$$f(0,\mathbf{y}) = g(\mathbf{y})$$
$$\frac{\partial f(x,\mathbf{y})}{\partial \ell} = A(x,\mathbf{y},h,f) \times f(x,\mathbf{y}) + B(x,\mathbf{y},h,f)$$

full linear schema deriving along ℓ captures **FP**

THE GLOBAL PICTURE (SO FAR) : FTC^0

MFCS 2025

$$f(0,\mathbf{y}) = g(\mathbf{y})$$

$$\frac{\partial f(x,\mathbf{y})}{\partial \ell} = A(x,\mathbf{y},h) \times f(x,\mathbf{y}) + B(x,\mathbf{y},h)$$

strict linear schema deriving along ℓ captures **FTC^0**

THE GLOBAL PICTURE (SO FAR) : $\mathbf{FTC}^0$

MFCS 2025

$$f(0, \mathbf{y}) = g(\mathbf{y})$$
$$\frac{\partial f(x, \mathbf{y})}{\partial \ell} = A(x, \mathbf{y}, h) \times f(x, \mathbf{y}) + B(x, \mathbf{y}, h)$$

REMARK (BINARY COUNTING). BCOUNT can be expressed by the IVP with initial value $f(0, \mathbf{y}) = g(\mathbf{y})$ and such that :

$$\frac{\partial f(x, \mathbf{y})}{\partial \ell} = k(x, \mathbf{y})$$

For $g(\mathbf{y}) = \mathrm{bit}(0, y)$ and $k(x, \mathbf{y}) = \mathrm{bit}(\ell(x), y)$, $f(x, x) = \#\{u : \mathrm{bit}(u, x) = 1\}$, which as desired outputs the sum of the bits of a given input.

THE GLOBAL PICTURE (SO FAR) : FTC^0

MFCS 2025

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$A = 0$ and $B = k(x,\mathbf{y}) \leq \{0,1\}$

For $g(\mathbf{y}) = \mathrm{bit}(0,y)$ and $k(x,\mathbf{y}) = \mathrm{bit}(\ell(x),y)$, f … $\}$, which as desired outputs the sum of the bits of a given input.

THE GLOBAL PICTURE (SO FAR) : FTC^0

MFCS 2025

$$f(0, \mathbf{y}) = g(\mathbf{y})$$
$$\frac{\partial f(x, \mathbf{y})}{\partial \ell} = A(x, \mathbf{y}, h) \times f(x, \mathbf{y}) + B(x, \mathbf{y}, h)$$

The solution of this system is : $f(x, \mathbf{y}) = \sum_{u=-1}^{\ell(x)-1} k(\alpha(u), \mathbf{y})$, where $\alpha(z) = 2^z - 1$ and $k(\alpha(-1), \cdot) = g(y)$.

REMARK (BINARY COUNTING). BCOUN $y) = g(\mathbf{y})$ and such that :

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For $g(\mathbf{y}) = \mathrm{bit}(0, y)$ and $k(x, \mathbf{y}) = \mathrm{bit}(\ell(x), y)$, $f(x, x) = \#\{u : \mathrm{bit}(u, x) = 1\}$, which as desired outputs the sum of the bits of a given input.

The global picture (so far) : FACC[n]

ICALP 2026

$k(x, \mathbf{y})$ takes values in $\{0, 1\}$

$$f(0, \mathbf{y}) = g(\mathbf{y})$$

$$\frac{\partial f(x, \mathbf{y})}{\partial \ell} = -n \times k(x, \mathbf{y}) \times \left\lfloor \frac{f(x, \mathbf{y})}{n-1} \right\rfloor + k(x, \mathbf{y})$$

strict schema, with special restricted “linearity” captures corresponding **FACC[n]**

THE GLOBAL PICTURE (SO FAR) : FACC[n]

for any $n \notin \{2, 6\}$, these are the very first recursion-theoretic characterizations of these classes

$$f(0, \mathbf{y}) = g(\mathbf{y})$$

$$\frac{\partial f(x, \mathbf{y})}{\partial \ell} = -n \times k(x, \mathbf{y}) \times \left\lfloor \frac{f(x, \mathbf{y})}{n-1} \right\rfloor + k(x, \mathbf{y})$$

THE GLOBAL PICTURE (SO FAR) : FACC[2]

ICALP 2026

$g(\mathbf{y}), k(x, \mathbf{y})$ take values in $\{0, 1\}$

$$f(0, \mathbf{y}) = g(\mathbf{y})$$

$$\frac{\partial f(x, \mathbf{y})}{\partial \ell} = -2 \times k(x, \mathbf{y}) \times f(x, \mathbf{y}) + k(x, \mathbf{y})$$

REMARK (COUNTING MODULO 2). Computation performed by MOD 2 gates can be simulated via ℓ-2ODE by considering $g(x, y) = 0$ and $k = \mathrm{bit}(\ell(x), y)$, i.e., the solution of the IVP s.t. cmod2(0,y)=0 and

$$\frac{\partial \mathrm{cmod2}(x, y)}{\partial \ell} = -2\mathrm{bit}(\ell(x), y) \times \mathrm{cmod2}(x, y) + \mathrm{bit}(\ell(x), y).$$

The count of input bits modulo 2 is obtained by computing cmod2(x, x).

Let us consider the input z of length $\ell(z) = 2$ and whose binary representation is $z_1 z_0$. Then,

$$
\begin{aligned}
\mathrm{cmod2}(0, z) &= 0 \\
\mathrm{cmod2}(1, z) &= \mathrm{cmod2}(0, z) - 2\mathrm{bit}(0, z) \times \mathrm{cmod2}(0, z) + \mathrm{bit}(0, z) = \mathrm{bit}(0, z) = z_0 \\
\mathrm{cmod2}(2, z) &= \mathrm{cmod2}(1, z) - 2\mathrm{bit}(1, z) \times \mathrm{cmod2}(1, z) + \mathrm{bit}(1, z) = z_0 - 2z_0 z_1 + z_1 \\
&= \begin{cases} 0 & \text{if } z_0 = z_1 \\ 1 & \text{otherwise.} \end{cases}
\end{aligned}
$$

$$\frac{\partial f(x, \mathbf{y})}{\partial \ell} = -2 \times k(x, \mathbf{y}) \times f(x, \mathbf{y}) + k(x, \mathbf{y})$$

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The count of input bits modulo 2 is obtained by computing cmod2(x, x).

THE GLOBAL PICTURE (SO FAR) : $\mathbf{FAC}^0$

MFCS 2025

$$f(0,\mathbf{y}) = g(\mathbf{y})$$

$$\frac{\partial f(x,\mathbf{y})}{\partial \ell} = f(x,\mathbf{y}) + k(x,\mathbf{y}) \quad k(x,\mathbf{y}) \leq 1$$

strict schema with restricted linearity s.t. $A = 1$ and $B \leq 1$) captures $\mathbf{FAC}^0$

THE GLOBAL PICTURE (SO FAR) : FNC[1]

MFCS 2025

$$f(0, \mathbf{y}) = g(\mathbf{y})$$

$$\frac{\partial f(x, \mathbf{y})}{\partial \ell} = -f(x, \mathbf{y}) + B(x, \mathbf{y}, f, h) \quad B \geq 0$$

non-strict (with simple calls) schema with restricted linearity, s.t. $A = -1$ and $B \geq 0$

THE GLOBAL PICTURE (SO FAR) : FACC[2]

MFCS 2025

$$f(0,\mathbf{y}) = g(\mathbf{y})$$

$$\frac{\partial f(x,\mathbf{y})}{\partial \ell} = -f(x,\mathbf{y}) + K(x,\mathbf{y},f,h) \quad K \leq 1$$

non-strict schema with restricted linearity s.t. $A = -1$ and $K \leq 1$

The global picture (so far)

	strict schemas along ℓ	non-strict schemas along ℓ	schemas along ℓ_2
FAC^0	$(k(x)-1)\times f(x)$ $f(x)+k(x)$	$(K(x,f)-1)\times f(x)$	$B(x)$
FACC[2]	$-k(x)\times f(x)+k(x)\times k'(x)$ $-2k(x)\times f(x)+k(x)$	$-f(x)+K(x,f)$	
FACC[n]	$-n\times k(x)\times \left\lfloor \frac{f(x)}{n-1} \right\rfloor + k(x)$		
FTC^0	$k(x)$ $A(x)\times f(x)+B(x)$	$K(x,f)$ $K(x,f)\times f(x)$	$B(x,f)$
FNC^1		$f(x)+K(x,f)$ $-f(x)+B(x,f)$	$A(x,f)\times f(x)+B(x,f)$
FAC^1		$B(x,f)$	

SUMMING UP

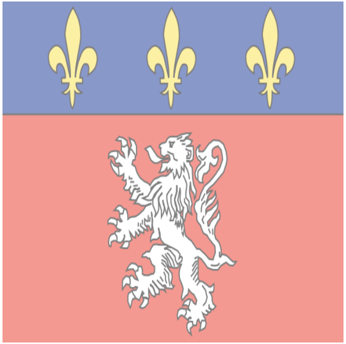
- new framework to study circuit complexity
- simple and uniform characterizations for small circuit classes ranging from $\mathbf{FAC}^0$ to $\mathbf{FNC}^1$
- first characterizations for classes including counting modulo *n* (and corresponding bounded arithmetic)

RESEARCH PERSPECTIVES

RESEARCH PERSPECTIVES

- ODE-based characterizations for $\mathbf{FAC}^i$ and $\mathbf{FNC}^i$ and for complexity hierarchies
- constraints on discrete ODE vs. circuit resources : (further) investigation into the computational power of schemas obtained deriving along ℓ_2
- ODE-based characterizations for circuit over the reals and arithmetic circuits
- ODE-based characterizations for counting classes (and for monotone ones)
- separation results via discrete ODEs
- logical and proof-theoretical counterparts

Thank you for your attention! Questions?



ADDITIONAL MATERIAL

From bounded recursion to ODEs

Definition (Discrete Derivative)

The *discrete derivative* of $\mathbf{f}(x)$ is defined as

$$\mathbf{f}'(x) = \mathbf{f}(x+1) - \mathbf{f}(x).$$

Definition (Discrete ODE)

System of equations of the form,

$$\frac{\partial \mathbf{f}(x, \mathbf{y})}{\partial x} = \mathbf{h}\big(\mathbf{f}(x, \mathbf{y}), x, \mathbf{y}\big).$$

From bounded recursion to ODEs

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$$\frac{\partial \mathbf{f}(x,\mathbf{y})}{\partial x} = \mathbf{f}(x+1,\mathbf{y}) - \mathbf{f}(x,\mathbf{y})$$

System of equations of the form,

$$\frac{\partial \mathbf{f}(x,\mathbf{y})}{\partial x} = \mathbf{h}\big(\mathbf{f}(x,\mathbf{y}), x, \mathbf{y}\big).$$

DISCRETE LINEAR ODE

Lemma (Solution of Linear ODE)

For matrices $\mathbf{A}$ and vectors $\mathbf{B}, \mathbf{G}$, the solution of

$$\begin{aligned} \mathbf{f}(0, \mathbf{y}) &= \mathbf{G}(\mathbf{y}) && \text{(initial condition)} \\ \mathbf{f}'(x, \mathbf{y}) &= \mathbf{A}(x, \mathbf{y}) \times \mathbf{f}(x, \mathbf{y}) + \mathbf{B}(x, \mathbf{y}) \end{aligned}$$

is

$$\mathbf{f}(x, \mathbf{y}) = \sum_{u=-1}^{x-1} \left(\prod_{t=u+1}^{x-1} \left(1 + \mathbf{A}(t, \mathbf{y})\right)\right) \times \mathbf{B}(u, \mathbf{y})$$

with conventions $\prod_{x}^{x-1} \kappa(x) = 1$ and $\mathbf{B}(-1, \mathbf{y}) = \mathbf{G}(\mathbf{y})$.

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algorithm unrolling the computation of $\mathbf{f}(x+1, \mathbf{y})$ from $\mathbf{f}(x, \mathbf{y}), \mathbf{f}(x-1, \mathbf{y}), \dots, \mathbf{f}(0, \mathbf{y})$

CHARACTERIZING $\mathbf{FTC}^0$

Lemma

If g and h are in $\mathbf{FTC^0}$ *and f is defined by ℓ-pODE from then, then f is in* $\mathbf{FTC^0}$.

Proof Sketch.

By definition of ℓ-ODE,

$$f(x,\mathbf{y}) = \sum_{u=-1}^{\ell(x)-1} \left(\prod_{t=u+1}^{\ell(x)-1} \Big(1 + A(r,\mathbf{y})\Big) \right) \times B(u,\mathbf{y})$$

with the convention that $\prod_{x}^{x-1} \kappa(x) = 1$ and $B(-1,\mathbf{y}) = g(\mathbf{y})$. Computing this expression can be done combining iterated addition and multiplication, which are in $\mathbf{FTC^0}$. □

CHARACTERIZING $\mathbf{FTC}^0$

EXAMPLE (BINARY COUNTING AND ITERATED ADDITION) BCOUNT can be expressed by the IVP with initial value $f(0,\mathbf{y}) = g(\mathbf{y})$ and such that :

$$\frac{\partial f(x,\mathbf{y})}{\partial \ell} = k(x,\mathbf{y})$$

For $g(y) = \mathrm{bit}(0,y)$ and $k(x,y) = \mathrm{bit}(\ell(x),y)$, $f(x,x) = \#\{u : \mathrm{bit}(u,x) = 1\}$, which as desired outputs the sum of the bits of a given input.
Similarly, ITADD is obtained considering an IVP such that :

$$\frac{\partial f(x,\mathbf{y})}{\partial \ell} = B(x,\mathbf{y},h(x,\mathbf{y})).$$

CHARACTERIZING FTC^0

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$A = 0$ and $B = k(x, \mathbf{y}) \in \{0, 1\}$

For $g(y) = \mathrm{bit}(0, y)$ and $k(x, y) =$ … $\{u : \mathrm{bit}(u, x) = 1\}$, which as desired outputs the sum of the bits of a given input.
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The solution of this system is : $f(x, \mathbf{y}) = \sum_{u=-1}^{\ell(x)-1} k(\alpha(u), \mathbf{y})$, where $\alpha(z) = 2^z - 1$ and $k(\alpha(-1), \cdot) = g(y)$.

Similarly, ITADD is obtained considering an IVP such that :

$$\frac{\partial f(x, \mathbf{y})}{\partial \ell} = B(x, \mathbf{y}, h(x, \mathbf{y})).$$

CHARACTERIZING FTC^0

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Similarly, ITADD is obtained con

We consider BCOUNT$(x) = f(x, x)$, where f is defined by ℓ-pODE so that $f(x, y) = \sum_{u=0}^{\ell(x)} \text{bit}(u, y)$.

$$\frac{\partial f(x, \mathbf{y})}{\partial \ell} = B(x, \mathbf{y}, h(x, \mathbf{y})).$$

CHARACTERIZING $\mathbf{FTC}^0$

Theorem (MFCS 2025)

$$\mathbf{FTC}^0 = [\mathbf{0}, \mathbf{1}, \mathrm{sg}, \ell, \mathrm{bit}, +, -, \div, \pi_i^p, \#; \circ, \ell\text{-pODE}]$$

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- $\supseteq$ Closure of ℓ-pODE.
- $\subseteq$ Relying on the previous Example and on the fact that ℓ-pODE encapsulates ℓ-ODE$_1$ computation.

On Clote's $\mathcal{N}_0'$

Definition (k Bounded Recursion on Notation, k-BRN)

For $h_i(x, \mathbf{y}) \leq k$ and $i \in \{0, 1\}$,

$$f(0, \mathbf{y}) = g(\mathbf{y})$$
$$f(\mathbf{s_i}(x), \mathbf{y}) = h_i(x, \mathbf{y}, f(x, \mathbf{y})).$$

Definition (Function Algebra $\mathcal{N}_0'$)

$$\mathcal{N}_0' = [\mathbf{0}, \pi_i^p, \mathbf{s}_0, \mathbf{s}_1, \mathbf{BIT}, \ell, \#; \circ, \mathrm{CRN}, 4\text{-BRN}]$$

On Clote's $\mathcal{N}_0'$

Definition (k Bounded Recursion on Notation, k-BRN)

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Theorem (Clote, 1993)

$$\mathbf{FNC^1} = \mathcal{N}_0'$$

CHARACTERIZING $\mathbf{FNC}^1$

CHARACTERIZING FNC[1]

Definition (Schema ℓ-bODE)

The function f is defined by ℓ-bODE from g and h if it is the solution of the IVP :

$$
\begin{aligned}
f(0, \mathbf{y}) &= g(\mathbf{y}) \\
\frac{\partial f(x, \mathbf{y})}{\partial \ell} &= -f(x, \mathbf{y}) + B(x, \mathbf{y}, h(x, \mathbf{y}), f(x, \mathbf{y}))
\end{aligned}
$$

where $B \geq 0$ is a sg-polynomial expression and f only occurs in it in expressions of the form $\mathrm{sg}(f(x, \mathbf{y}) - c)$, for any constant c.

CHARACTERIZING FNC[1]

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$A = -1$

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CHARACTERIZING $\mathbf{FNC}^1$

$$[\mathbf{0}, \mathbf{1}, \ell, \mathrm{sg}, \mathbf{BIT}, +, -, \div 2, \#, \pi_i^p; \circ, \ell\text{-pODE}, \ell\text{-bODE}]$$

CHARACTERIZING $\mathbf{FNC}^1$

Theorem (MFCS 2025)

$$\mathbf{FNC}^1 = [0, 1, \ell, \mathrm{sg}, \mathrm{bit}, +, -, \div 2, \#, \pi_i^p; \circ, \ell\text{-pODE}, \ell\text{-bODE}]$$

CHARACTERIZING FNC^1

Theorem (MFCS 2025)

$$\mathbf{FNC}^1 = [0, 1, \ell, \mathrm{sg}, \mathrm{bit}, +, -, \div 2, \#, \pi^p_i; \circ, \ell\text{-pODE}, \ell\text{-bODE}]$$

- $\supseteq$ Closure property of ℓ-pODE and ℓ-bODE.
- $\subseteq$ Indirect proof passing through Clote's $\mathcal{N}'_0$ and rewriting the defining 4-BRN schema via ℓ-bODE.

Characterizing FNC^1

Theorem (MFCS 2025)

$$\mathbf{FNC^1} = [0, 1, \ell, \mathrm{sg}, \mathrm{bit}, +, -, \div 2, \#, \pi_i^p; \circ, \ell\text{-pODE}, \ell\text{-bODE}]$$

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$[0, \pi_i^p, s_0, s_1, \ell, \mathrm{bit}, \#; \circ, \mathrm{CRN}, \text{4-BRN}]$

CHARACTERIZING FNC^1

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$$\mathbf{FNC^1} = [0, 1, \ell, \mathrm{sg}, \mathrm{bit}, +, -, \div 2, \#, \pi_i^p; \circ, \ell\text{-pODE}, \ell\text{-bODE}]$$

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$[0, \pi_i^p, s_0, s_1, \ell, \mathrm{bit}, \#; \circ, \mathrm{CRN}, \text{4-BRN}]$

CHARACTERIZING FNC[1]

Remark (4-BRN via ℓ-bODE)

Given f defined by 4-BRN from g and h_i's, we consider the solution of the IVP below

$$
\begin{aligned}
\mathbf{4brn}(x, \mathbf{y}) &= g(\mathbf{y}) \\
\frac{\partial \mathbf{4brn}(x, \mathbf{y})}{\partial \ell} &= -\mathbf{4brn}(x, \mathbf{y}) + h_{z(x)}(\ell(x), \mathbf{y}, 0) \times \mathrm{cosg}(\mathbf{4brn}(x, \mathbf{y})) \\
&\quad + h_{z(x)}(\ell(x), \mathbf{y}, 1) \times \mathrm{sg}(\mathbf{4brn}(x, \mathbf{y})) \times \mathrm{cosg}(\mathbf{4brn}(x, \mathbf{y}) - 1) \\
&\quad + \dots \\
&\quad + h_{z(x)}(\ell(x), \mathbf{y}, 4) \times \mathrm{sg}(\mathbf{4brn}(x, \mathbf{y}) - 3) \times \mathrm{cosg}(\mathbf{4brn}(x, \mathbf{y}) - 4)
\end{aligned}
$$

with $z(x) = \mathbf{BIT}(\ell(y) - \ell(x) - 1, y)$. This is an instance of ℓ-bODE. Then

$$\mathbf{4brn}(x, x, \mathbf{z}).$$

CHARACTERIZING FNC^1

Remark (4-BRN via ℓ-bODE)

Given f defined by 4-BRN from g and h_i's, we consider the solution of the IVP below

$\mathbf{4brn}(x, \mathbf{y})$ only occurs B only in expressions of the form $\mathrm{sg}(\mathbf{4brn}(x, \mathbf{y}) - c)$

$$
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&\quad + h_{z(x)}(\ell(x), \mathbf{y}, 1) \times \mathrm{sg}(\mathbf{4brn}(x, \mathbf{y}) \times \mathrm{cosg}(\mathbf{4brn}(x, \mathbf{y}) - 1) \\
&\quad + \dots \\
&\quad + h_{z(x)}(\ell(x), \mathbf{y}, 4) \times \mathrm{sg}(\mathbf{4brn}(x, \mathbf{y}) - 3) \times \mathrm{cosg}(\mathbf{4brn}(x, \mathbf{y}) - 4)
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with $z(x) = \mathbf{BIT}(\ell(y) - \ell(x) - 1, y)$. This is an instance of ℓ-bODE. Then

$$\mathbf{4brn}(x, x, \mathbf{z}).$$

CHARACTERIZING FACC[2]

Theorem (MFCS 2025)

$$\mathbf{FACC[2]} = [0, 1, \mathrm{sg}, \ell, +, -, \div 2, \pi_i^p, \#; \circ, \ell\text{-ODE}_1, \ell\text{-ODE}_3, \ell\text{-b}_0\text{ODE}]$$

CHARACTERIZING FACC[2]

To count the number of 1's in z we consider the solution of the following system

$$\begin{aligned}\mathbf{par}(0,y) &= \mathrm{bit}(0,y)\\ \frac{\partial \mathbf{par}(x,y)}{\partial \ell} &= -\mathbf{par}(x,y) + (\mathrm{sg}(\mathbf{par}(x,y)\times \mathrm{cosg}(\mathrm{bit}(\ell(x)+1,y))\\ &\qquad + \mathrm{cosg}(\mathbf{par}(x,y))\times \mathrm{sg}(\mathrm{bit}(\ell(x)+1,y)))\end{aligned}$$

which is clearly an instance of ℓ-b$_0$ODE and compute $\mathbf{par}(z,z)$.

The

, ℓ-b$_0$ODE]

- $\supseteq$ Basic functions are computable [...] FACC[2] and closure under composition and ℓ-ODE$_1$, ℓ-ODE$_3$ and ℓ-b$_0$ODE.
- $\subseteq$ Follows from the fact that the parity function can be rewritten via ℓ-b$_0$ODE.